1. (*8 marks*) Consider the function
$$f(x) = ax^2 + \frac{b}{x}$$
where a and b are constants. Find $f'(x)$ using the definition of the derivative (Newton's quotient).

Solution

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} && \text{Definition of the derivative}\\
&= \lim_{h\to 0} \frac{\left[a(x+h)^2 + \frac{b}{x+h}\right] - \left[ax^2 + \frac{b}{x}\right]}{h} && \text{Application of the definition}\\
&= \lim_{h\to 0} \frac{1}{h}\left[a(x+h)^2 - ax^2\right] + \lim_{h\to 0} \frac{1}{h}\left[\frac{b}{x+h} - \frac{b}{x}\right] && \text{Distribution of limits}\\
&= \lim_{h\to 0} \frac{1}{h}\left[ax^2 + 2axh + ah^2 - ax^2\right] + \lim_{h\to 0} \frac{1}{h}\left[\frac{bx - b(x+h)}{x(x+h)}\right] && \text{Common denominator}\\
&= \lim_{h\to 0} \frac{1}{\cancel{h}}\left[\cancel{ax^2} + 2ax\cancel{h} + ah^{\cancel{2}} - \cancel{ax^2}\right] + \lim_{h\to 0} \frac{1}{\cancel{h}}\left[\frac{\cancel{bx} - \cancel{bx} - b\cancel{h})}{x(x+h)}\right] && \text{Simplification}\\
&= \lim_{h\to 0} \left[2ax + \cancelto{0}{ah}\right] + \lim_{h\to 0} \left[\frac{-b}{x(x+\cancelto{0}{h})}\right] && \text{Evaluation of limits}\\
&= 2ax - \frac{b}{x^2}
\end{aligned}$$

2. (*8 marks*) Determine the values of a and b so that $\lim_{x\to\infty} \sqrt{x^2-5x+1} - ax + b = 0$

Solution

$$\begin{aligned}
&\lim_{x\to\infty} \sqrt{x^2-5x+1} - (ax-b) && \text{Rewriting the expression}\\
&= \lim_{x\to\infty} \left(\sqrt{x^2-5x+1} - (ax-b)\right) \cdot \frac{(\sqrt{x^2-5x+1} + (ax-b))}{(\sqrt{x^2-5x+1} + (ax-b))} && \text{Conjugate}\\
&= \lim_{x\to\infty} \frac{x^2-5x+1-(ax-b)^2}{\sqrt{x^2-5x+1} + (ax-b)}\\
&= \lim_{x\to\infty} \frac{x^2 - a^2x^2 + 2abx - 5x + 1 - b^2}{|x|\sqrt{1-\frac{5}{x}+\frac{1}{x^2}} + x(a-\frac{b}{x})} && \text{Simplifying denominator}\\
&= \lim_{x\to\infty} \frac{(1-a^2)x^2 + (2ab-5)x + 1 - b^2}{+x\sqrt{1-\frac{5}{x}+\frac{1}{x^2}} + x(a-\frac{b}{x})} && \text{Grouping similar terms together}
\end{aligned}$$

The dominant power in the numerator, x^2, must cancel: $\Rightarrow 1 - a^2 = 0 \quad \Rightarrow \quad a = \pm 1$

The top power, x must cancel again, so $a = +1$ (otherwise the limit would be ∞; not 0).

$$\lim_{x\to\infty} \frac{(2ab-5)x + 1 - b^2}{+x\sqrt{1 - \frac{5}{x} + \frac{1}{x^2}} + x(1 - \frac{b}{x})} 0$$

$$\lim_{x\to\infty} \frac{(2b-5)x + 1 - b^2}{+x\sqrt{1 - \frac{5}{x} + \frac{1}{x^2}} + x(1 - \frac{b}{x})} = 0$$

$$\lim_{x\to\infty} \frac{(2b-5) + \cancelto{0}{\frac{1}{x}} - \cancelto{0}{\frac{1}{b^2}}}{\sqrt{1 - \cancelto{0}{\frac{5}{x}} + \cancelto{0}{\frac{1}{x^2}}} + (1 - \cancelto{0}{\frac{b}{x}})} = 0 \quad \Rightarrow \quad 2b - 5 = 0 \quad \Rightarrow \quad b = \frac{5}{2}$$

$$\therefore \lim_{x\to\infty} \sqrt{x^2 - 5x + 1} - x + \frac{5}{2} = 0$$

3. (*12 marks*) Evaluate the following limits and simplify your answers. You may use L'Hopital's rule where appropriate.

Solution

(a) $\lim_{x\to 0^+} (\sin x)^{\tan x} = 0^0$ — Indeterminate power

$$\begin{aligned}
y &= (\sin x)^{\tan x} && \text{Take the ln of both sides}\\
\lim_{x\to 0^+} \ln y &= \lim_{x\to 0^+} \tan x \cdot \ln(\sin x) = 0 \cdot (-\infty) && \text{Indeterminate product}\\
&= \lim_{x\to 0^+} \frac{\ln(\sin x)}{\cot x} = -\frac{\infty}{\infty} && \text{Apply L'Hopital's Rule}\\
&\overset{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{\sin x} \cdot \cos x}{-\csc^2 x}\\
&= \lim_{x\to 0^+} \frac{\cos x}{\sin x} \cdot \frac{\sin^2 x}{-1} && \text{Simplify}\\
&= \lim_{x\to 0^+} -\cos x \cdot \sin x\\
&= -(1)(0) && \text{Evaluate the limit}\\
\lim_{x\to 0^+} \ln y &= 0 && \text{Solve for } y\\
y &= e^0 = 1
\end{aligned}$$

(b) $\lim_{t\to 1} \dfrac{nt^{n+1}-(n+1)t^n+1}{(t-1)^2}$ for $n \geq 2$

$$\begin{aligned}
&\lim_{t\to 1}\frac{n\,t^{n+1}-(n+1)\,t^n+1}{(t-1)^2} = \frac{0}{0} \\
&\stackrel{H}{=} \lim_{t\to 1}\frac{n(n+1)\,t^n-(n+1)(n)\,t^{n-1}}{2(t-1)} = \frac{0}{0} && \text{L'Hopital's Rule}\\
&\stackrel{H}{=} \lim_{t\to 1}\frac{n(n+1)(n)\,t^{n-1}-(n+1)(n)(n-1)\,t^{n-2}}{2} = \frac{0}{0} && \text{L'Hopital's Rule}\\
&= \lim_{t\to 1}\frac{n(n+1)(n)\,1^{n-1}-(n+1)(n)(n-1)\,1^{n-2}}{2} && \text{Evaluation of limit}\\
&= \lim_{t\to 1}\frac{n(n+1)(n)\,\cancelto{1}{1^{n-1}}-(n+1)(n)(n-1)\,\cancelto{1}{1^{n-2}}}{2} && \text{Simplifcation}\\
&= \frac{n(n+1)[n-(n-1)]}{2}\\
&= \frac{n(n+1)}{2}
\end{aligned}$$

(c) $\lim_{x\to 0^+} \sqrt[3]{x^2+x}\,\sin\left(\dfrac{1}{x^2}\right)$

Squeeze Theorem: The function $\sin\left(\frac{1}{x^2}\right)$ is sandwiched between ± 1. Therefore,

$$\begin{aligned}
&-1 \leq \sin\left(\frac{1}{x^2}\right) \leq 1\\
&-\sqrt[3]{x^2+x} \leq \sqrt[3]{x^2+x}\cdot\sin\left(\frac{1}{x^2}\right) \leq \sqrt[3]{x^2+x}\\
&-\lim_{x\to 0^+}\sqrt[3]{x^2+x} \leq \lim_{x\to 0^+}\sqrt[3]{x^2+x}\,\sin\left(\frac{1}{x^2}\right) \leq \lim_{x\to 0^+}\sqrt[3]{x^2+x}
\end{aligned}$$

$$\left.\begin{aligned}
\because\ &\lim_{x\to 0^+} -\sqrt[3]{x^2+x} = 0\\
\&\ &\lim_{x\to 0^+} +\sqrt[3]{x^2+x} = 0
\end{aligned}\right\} \Rightarrow \quad \lim_{x\to 0^+}\sqrt[3]{x^2+x}\,\sin\left(\frac{1}{x^2}\right) = 0 \text{ by Squeeze Theorem.}$$

4. (*8 marks*) A container in the shape of a right circular cone with vertex angle a right angle is partially filled with water.

Solution

(a) Suppose water is added at the rate of $3\,cm^3/s$. How fast is the water level rising when the height $h = 2\,cm$?

Let $r =$ the radius of the cone
$h =$ the height of the cone

We have: $\dfrac{dV}{dt} = 3\,cm^3/s$
We want: $\dfrac{dh}{dt}$ when $h = 2\,cm$

The volume of a cone is $V = \frac{1}{3}\pi r^2 h$ $\because$ the cone is right-circular, $r = h$

$$\begin{aligned}
\therefore\ V(t) &= \frac{1}{3}\pi[h(t)]^3 && \text{Related rates equation}\\
V'(t) &= \frac{1}{3}\pi\cdot 3[h(t)]^2\cdot h'(t) && \text{Differentiating wrt time}\\
\frac{dV}{dt} &= \pi h^2\cdot\frac{dh}{dt} && \text{Leibniz notation}\\
3 &= \pi(2)^2\cdot\frac{dh}{dt} && \text{Calculating the rate}\\
\frac{dh}{dt} &= \frac{3}{4\pi}\,cm/s
\end{aligned}$$

(b) Suppose instead no water is added, but water is being lost by evaporation. Show that the level falls at a constant rate.

Assume that the rate of evaporation is proportional to the surface area of the liquid. In other words,

$$\frac{dV}{dt} = k\pi r^2 = k\pi h^2$$

for some constant k that is negative.
Now,

$$\begin{aligned}
\frac{dv}{dt} &= \pi h^2\cdot\frac{dh}{dt} && \text{from part (a)}\\
k\pi h^2 &= \pi h^2\cdot\frac{dh}{dt}\\
k\cancel{\pi}\cancel{h^2} &= \cancel{\pi}\cancel{h^2}\cdot\frac{dh}{dt}\\
k &= \frac{dh}{dt}
\end{aligned}$$

$\because\ \dfrac{dh}{dt} = k \quad\Rightarrow\quad$ the height of the liquid is decreasing at a constant rate.

5. (*8 marks*) Consider the function

$$f(x) = \begin{cases} \dfrac{\sin^2 x}{x} & ; \quad x < 0 \\ \ln(1 + a\tan x + b\cos x) & ; \quad x \geq 0 \end{cases}$$

Determine the values of the constants a and b that will make the function $f(x)$ continuous and differentiable on the interval $(-1, 1)$.

Solution

The only possible discontinuity is at $x = 0$.

Continuity

$f(0) = \ln(1 + b)$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin^2 x}{x} \overset{H}{=} \lim_{x\to 0^-} \frac{2\sin x \cdot \cos x}{1} = 0$$

$\ln(1 + b) = 0 \quad \Rightarrow \quad b = 0$

Differentiability

For $x < 0$

$$\begin{aligned} f'(x) &= \frac{2\sin x \cdot \cos x \cdot x - \sin^2 x}{x^2} && \text{Quotient rule} \\ \lim_{x\to 0^-} f'(x) &= \lim_{x\to 0^-} \frac{2\sin x \cdot \cos x \cdot x - \sin^2 x}{x^2} && \text{Evaluate limit} \\ &= \lim_{x\to 0^-} \frac{2\sin x \cdot \cos x}{x} - \frac{\sin^2 x}{x^2} \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

For $x > 0$

$$\begin{aligned} f'(x) &= \frac{a\sec^2 x}{1 + a\tan x} && \text{Quotient rule} \\ \lim_{x\to 0^+} f'(x) &= \lim_{x\to 0^+} \frac{a\sec^2 x}{1 + a\tan x} && \text{Evaluate limit} \\ &= a \end{aligned}$$

$\because \lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x) \quad \Rightarrow \quad a = 1$

$\therefore$ For $f(x)$ to be continuous and differentiable on $(-1, 1)$ $a = 1$ and $b = 0$

6. (*8 marks*) The equation

$$x^5 + x^2y + y^3 = 4y + 3$$

defines y implicitly as a function of x near the point $(x, y) = (1, 2)$.

Solution

(a) Determine the value of y' at $(x, y) = (1, 2)$

$$\begin{aligned} x^5 + x^2y + y^3 &= 4y + 3 \\ 5x^4 + 2x \cdot y + x^2 \cdot y' + 3y^2 \cdot y' &= 4 \cdot y' && \text{Implicit differentiation} \\ y' &= \frac{5x^4 + 2xy}{4 - x^2 - 3y^2} && \text{Isolating for } y' \end{aligned}$$

$$y'(1, 2) = \frac{5 + 4}{4 - 1 - 12} = -1$$

(b) Determine the value of y'' at $(x, y) = (1, 2)$

Implicit differentiation on: $5x^4 + 2x \cdot y + x^2 \cdot y' + 3y^2 \cdot y' = 4 \cdot y'$

$$\begin{aligned} 20x^3 + 2 \cdot y + 2x \cdot y' + 2x \cdot y' + x^2 \cdot y'' + 6y \cdot y' \cdot y' + 3y^2 \cdot y'' &= 4 \cdot y'' \\ 20x^3 + 2 \cdot y + 4x \cdot y' + x^2 \cdot y'' + 6y \cdot (y')^2 + 3y^2 \cdot y'' &= 4 \cdot y'' \\ \frac{20x^3 + 2y + 4xy' + 6y(y')^2}{4 - x^2 - 3y^2} &= y'' \end{aligned}$$

$$y''(1, 2) = \frac{20 + 4 - 4 + 12}{4 - 1 - 12} = -\frac{32}{9}$$

(c) Estimate the value of y when $x = 0.97$ by using linear approximations/differentials and the information from part (a).

Linear Approximations with $a = 1$ and $x = 0.97$

$$\begin{aligned} L(x) &= f(a) + f'(a) \cdot (x - a) \\ &= f(1) + f'(1) \cdot (x - 1) \\ L(x) &= 2 - 1 \cdot (x - 1) \end{aligned}$$

$$\begin{aligned} L(0.97) &= 2 - (0.97 - 1) \\ &= 2.03 \end{aligned}$$

7. (*8 marks*) Let $f(x) = |x^2 - 5x - 6| + 2x^2 + 17x$. Find the absolute maximum and absolute minimum of $f(x)$ for $-6 \le x \le 3$ and the locations where they occur.

Solution

Observe that $f(x) = |x^2 - 5x - 6| + 2x^2 + 17x = |(x-6)(x+1)| + 2x^2 + 17x$.

Rewriting $f(x)$ as a piecewise defined function over the given interval:

$$f(x) = \begin{cases} +(x-6)(x+1) + 2x^2 + 17 & ; \quad -6 < x \le -1 \\ -(x-6)(x+1) + 2x^2 + 17 & ; \quad -1 < x \le 3 \end{cases}$$

$$f(x) = \begin{cases} 3x^2 + 12x - 6 & ; \quad -6 < x \le -1 \\ x^2 + 22x + 6 & ; \quad -1 < x \le 3 \end{cases}$$

$$f'(x) = \begin{cases} 6x + 12 & ; \quad -6 < x \le -1 \\ 2x + 22 & ; \quad -1 < x \le 3 \end{cases}$$

$f'(x) = 0$ when

$$\begin{aligned} 6x + 12 = 0 \quad &\Rightarrow \quad x = -2 \\ 2x + 22 = 0 \quad &\Rightarrow \quad x = -11\text{; not in the interval.} \end{aligned}$$

$f'(-1)$ - DNE, so $x = -1$ is singular.

Testing the endpoints and the critical values:

$$\begin{aligned} &f(-6) = 30 \\ &f(-2) = -18 \quad \leftarrow \text{absolute minimum.} \\ &f(-1) = -15 \\ &f(3) = 81 \quad \leftarrow \text{absolute maximum.} \end{aligned}$$

8. (*8 marks*) A tall, open pot has the shape of a cylinder, with a circular base of radius a inches. A cabbage with radius r inches, where $0 < r < a$, is placed in the pot, and the pot is filled with just enough water to cover the cabbage completely. What is the radius of the cabbage which requires the greatest amount of water to accomplish this?

Solution

Let $a =$ the radius of the pot.

$r =$ the radius of the cabbage.

The water in the pot will form a cylindrical column. In order to cover the cabbage completely, the height of the water column must be $h = 2 \cdot r$.

The volume of water needed to accomplish this is:

$$V_{Water} = V_{Cylindrical\,Column} - V_{Cabbage}$$

$$V(r) = 2\pi a^2 \cdot r - \frac{4}{3}\pi r^3 \qquad 0 < r < a \qquad \text{Open interval}$$

$$V'(r) = 2\pi a^2 - 4\pi r^2$$

$V'(r) = 0$ when $r = \pm\sqrt{\frac{a^2}{2}} \quad \Rightarrow \quad r = \dfrac{a}{\sqrt{2}}$

Derivative check:

When: $0 < r < \frac{a}{\sqrt{2}}$ $\qquad V'(r) > 0$

When: $\frac{a}{\sqrt{2}} < r < a$ $\qquad V'(r) < 0$

$\Rightarrow V(r)$ has a absolute maximum at $r = \dfrac{a}{\sqrt{2}}$

$\therefore$ If the pot has radius a, then a cabbage with radius $r = \frac{a}{\sqrt{2}}$ inches, will require the most water to cover it completely.

9. (*12 marks*) Consider the function

$$f(x) = \frac{ax^2}{x^2 + b^2},$$

where a is a positive constant and b is a constant.

Solution

(a) Domain: $D_f = \mathbb{R}$

(b) Asymptotes:

Vertical: None $\because D_f = \mathbb{R}$

Horizontal: $\displaystyle\lim_{x\to\pm\infty} \frac{ax^2}{x^2+b^2} \overset{H}{=} \lim_{x\to\pm\infty} \frac{a \cdot \cancel{2x}}{\cancel{2x}} = a \quad \Rightarrow \quad y = a$ is a HA

(c) Intervals of increasing/decreasing:

$$\begin{aligned} f'(x) &= \frac{(x^2+b^2)\cdot a2x - ax^2 \cdot (2x)}{(x^2+b^2)^2} \qquad \text{Quotient rule} \\ &= \frac{2ax^3 + 2abx - 2ax^3}{(x^2+b^2)^2} \\ f'(x) &= \frac{2abx}{(x^2+b^2)^2} \end{aligned}$$

$f'(x) = 0$ when $x = 0$
$f'(x) = dne$ never

Sign chart for $f'(x)$

	0	
$2abx$	$-$	$+$
$(x^2 + b^2)$	$+$	$+$
$f'(x)$	$-$	$+$
$f(x)$	$\searrow$	$\nearrow$

$\Rightarrow$ f is decreasing on $(-\infty, 0)$
f is increasing on $(0, \infty)$

(d) Intervals of concavity:

$$f''(x) = \frac{(x^2+b^2)^2 \cdot 2ab - 2abx \cdot 2(x^2+b^2) \cdot 2x}{[(x^2+b^2)^2]^2} \qquad \text{Quotient rule \& chain rule}$$

$$= \frac{2ab\cancel{(x^2+b^2)} \cdot [(x^2+b^2) - 2 \cdot 2x]}{(x^2+b^2)^{\cancel{4}3}}$$

$$f''(x) = \frac{2ab \cdot (b^2 - 3x^2)}{(x^2+b^2)^3}$$

$f''(x) = 0$ when $b^2 - 3x^2 = 0 \quad \Rightarrow \quad x \pm \sqrt{\frac{b^2}{3}}$
$f''(x) = dne$ never

Sign chart for $f''(x)$

	$-\sqrt{\frac{b^2}{3}}$	$+\sqrt{\frac{b^2}{3}}$	
$2ab(b^2 - 3x^2)$	$-$	$+$	$-$
$(x^2 + b^2)$	$+$	$+$	$+$
$f''(x)$	$-$	$+$	$-$
$f(x)$	$\cap$	$\cup$	$\cap$

$$\Rightarrow f \text{ is concave down on } \left(-\infty, -\sqrt{\tfrac{b^2}{3}}\right) \cup \left(+\sqrt{\tfrac{b^2}{3}}, \infty\right)$$

$$f \text{ is concave up on } \left(-\sqrt{\tfrac{b^2}{3}}, +\sqrt{\tfrac{b^2}{3}}\right)$$

(e) Extrema: $(0, 0)$ is an absolute minimum.

(f) Identification of inflection points: $\left(-\sqrt{\frac{b^2}{3}}, \frac{a}{4}\right)$ and $\left(\sqrt{\frac{b^2}{3}}, \frac{a}{4}\right)$ are inflection points.

10. (*8 marks*) Assume that $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b). Also assume that $f(a)$ and $f(b)$ have opposite signs and $f'(x) \neq 0$ between a and b. Show that $f(x) = 0$ exactly once between a and b.

Solution

By the IVT $f(x) = 0$ on (a, b) at least once. Assume for contradiction that $f(x_1) = 0 = f(x_2)$ for $x_1, x_2 \in (a, b)$ with $x_1 \neq x_2$. Then by MVT $\exists c \in (x_1, x_2)$ such that

$$f'(x) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0$$

But $f'(x) \neq 0$. Contradiction. There is only one root. *QED*

11. (*12 marks*) In each part below, use the method of your choice (that works) to find the derivative. You do not have to simplify your solution.

Solution

(a) Determine $f'(x)$ if $f(x) = \sqrt{1 - \sqrt{1 - \sqrt{1 - x^4}}}$

The derivative of $f(x)$ is obtained by applying the Chain Rule four times.

$$\begin{aligned} f(x) &= (\,1 - (\,1 - (\,1 - x^4\,)^{\,1/2})^{\,1/2})^{\,1/2} \\ f'(x) &= \frac{1}{2}\left(1 - (\,1 - (\,1 - x^4\,)^{\,1/2})^{\,1/2}\right)^{-1/2} \cdot \\ &\qquad \left(0 - \frac{1}{2}(1 - (1 - x^4)^{\,1/2})^{-1/2}\right) \cdot \left(0 - \frac{1}{2}(1 - x^4)^{-1/2}\right) \cdot (-4x^3) \end{aligned}$$

(b) Determine $g'(x)$ if $g(x) = \dfrac{e^x \cdot 5^{x^2} \cdot 11^x}{4^{x+8} \cdot 3^{x^3}}$

The derivative of $g(x)$ is computed using logarithmic differentiation.

$$\begin{aligned}
y &= \frac{e^x \cdot 5^{x^2} \cdot 11^x}{4^{x+8} \cdot 3^{x^3}} \\
\ln y &= \ln\left[\frac{e^x \cdot 5^{x^2} \cdot 11^x}{4^{x+8} \cdot 3^{x^3}}\right] \\
&= x + x^2 \ln 5 + x \ln 11 - [(x+8)\ln 4 + x^3 \ln 3] \\
\frac{1}{y} \cdot y' &= 1 + 2x \ln 5 + \ln 11 - [\ln 4 + 3x^2 \ln 3] \\
y' &= y \cdot \left[1 + 2x \ln 5 + \ln 11 - \ln 4 - 3x^2 \ln 3\right] \\
&= \frac{e^x \cdot 5^{x^2} \cdot 11^x}{4^{x+8} \cdot 3^{x^3}} \left[1 + 2x \ln 5 + \ln 11 - \ln 4 - 3x^2 \ln 3\right]
\end{aligned}$$

(c) Determine $\dfrac{d}{dx}(f^{-1})(2)$ if $f(x) = x + x^3$

The derivative of the inverse function, is obtained via the formula

$$\frac{d}{dx}(f^{-1})(2) = \frac{1}{f'(f^{-1}(2))}$$

$f(x) = x + x^3$

By inspection, $f(1) = 2 \quad \Rightarrow \quad f^{-1}(2) = 1$

$f'(x) = 1 + 3x^2$

$$\therefore \frac{d}{dx}(f^{-1})(2) = \frac{1}{f'(1)} = \frac{1}{1 + 3(1)^2} = \frac{1}{4}$$