Double Pendulum System

Ariel Halimi (2078496)
Vlad-Andrei Stelea (2050830)

201-HTL-VA Differential Equations section 00001

May 20, 2022

Contents

1. Abstract

The double pendulum is a system with remarkable non-linear behaviour. It became a popular tool in academic contexts for demonstrating the nuanced reaction of an apparently simple physical equipment or validating techniques for researching nonlinear processes. Furthermore, the double pendulum is frequently employed in a variety of modelling applications such as robotics and human movement study. Surprisingly, there is an absence of properly documented hardware that allows for the design, construction, and reliable tracking and collection of data from a double pendulum. This study will provide detailed study of the double pendulum system. It includes an in-depth look at the mathematical process of determining its motion as well as multiple computer simulations that showcase its chaotic nature.

French translation:

Le double pendule est un système au comportement non linéaire remarquable. Il est devenu un outil populaire dans les contextes académiques pour démontrer la réaction nuancée d'un équipement physique apparemment simple ou pour valider des techniques de recherche de processus non linéaires. De plus, le double pendule est fréquemment utilisé dans une variété d'applications de modélisation telles que la robotique et l'étude du mouvement humain. Étonnamment, il n'y a pas de matériel correctement documenté qui permette la conception, la construction, le suivi et la collecte fiables de données à partir d'un double pendule. Cette étude fournira une étude détaillée du système à double pendule. Il comprend un examen approfondi du processus mathématique de détermination de son mouvement ainsi que de multiples simulations informatiques qui mettent en valeur sa nature chaotique.

2. Introduction

2.1. Brief description of a double pendulum system

A double pendulum is unquestionably a natural wonder. The increase in complexity found while transitioning from a basic pendulum to a double pendulum is astounding. A basic pendulum's oscillations are regular. These oscillations are harmonic for tiny departures from equilibrium and may be characterised by a sine or cosine function. The period of nonlinear oscillations depends on the amplitude, but the motion remains regular. In other words, for a simple pendulum, the approximation of tiny oscillations completely represents the system's key features.

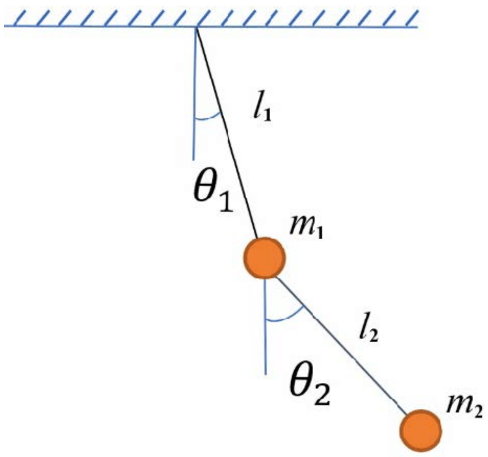
The "behaviour" of a double pendulum is significantly different. The double pendulum exhibits the phenomenon of beats in the regime of tiny vibrations. The nature of pendulum oscillations changes dramatically as energy increases; the oscillations become chaotic. Despite the fact that the double pendulum may be represented by a system of four ordinary differential equations, which is by a totally deterministic model, the occurrence of chaos appears to be remarkable. This characteristic is similar to the Lorenz system, in which a deterministic model with three equations exhibits chaotic behaviour. We can analyse its movement using the python simulation provided at the end of this paper.

3. Numerical experimentation

3.1. Equation of motion

3.1.1. Kinematics

The kinematics of a double pendulum hold a critical importance in describing its chaotic motion. When trying to find the equation of motion, only the position, and velocity are being considered in terms of the variables that specify the state of this system. However, it is important to notice that kinematics does not regard the forces involved. Let us look at the following image of this complex system and introduce some variables. However, in this section (3.1.) we will study the pendulum rods as being massless and rigid and the masses as being point masses.



$x = horizontal\ position\ of\ the\ pendul\ mass$

$y = vertical\ position\ of\ pendul\ mass$

$\theta = angle\ of\ pendul$

$l = length\ of\ rod\ (stays\ constant)$

Figure 1: double pendulum system

we consider the origin to be at the pivot point of the upper pendulum and y as increasing upwards. Let's begin by writing expressions for each positions (x_1, y_1, x_2, y_2) by using simple trigonometry while considering both angles (θ_1, θ_2).

$$x_1 = l_1 \sin(\theta_1) \quad (1)$$

$$y_1 = -l_1 \cos(\theta_1) \quad (2)$$

$$x_2 = x_1 + l_2 \sin(\theta_2) \quad (3)$$

$$y_2 = y_1 - l_2 \cos(\theta_2) \quad (4)$$

After determining the above equations of positions, we can deduce the equation for velocity by finding the derivative of the position with respect to time.

$$x_1' = \theta_1' l_1 \cos(\theta_1) \tag{5}$$

$$y_1' = \theta_1' l_1 \sin(\theta_1) \tag{6}$$

$$x_2' = x_1' + \theta_2' l_2 \cos(\theta_2) \tag{7}$$

$$y_2' = y_1' + \theta_2' l_2 \sin(\theta_2) \tag{8}$$

Now, to find the acceleration equations we can find the second derivative of the position equations or the derivative of the velocity equations. Either way, we will get the same answer.

$$x_1'' = -\left(\theta_1'\right)^2 l_1 \sin(\theta_1) + \theta_1'' l_1 \cos(\theta_1) \tag{9}$$

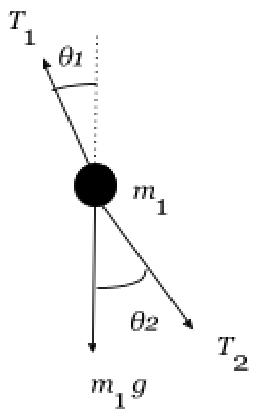
$$y_1'' = \left(\theta_1'\right)^2 l_1 \cos(\theta_1) + \theta_1'' l_1 l_1 \sin(\theta_1) \tag{10}$$

$$x_2'' = x_1'' - \left(\theta_2'\right)^2 l_2 \sin(\theta_2) + \theta_2'' l_2 \cos(\theta_2) \tag{11}$$

$$y_2'' = y_1'' + \left(\theta_2'\right)^2 l_2 \cos(\theta_2) + \theta_2'' l_2 \sin(\theta_2) \tag{12}$$

3.1.2. Forces in the double pendulum system

In this section, let's treat both pendulum masses as point particles and draw their free body diagrams. But first let's define some variables.



$$\begin{aligned} T &= \textit{tension in the rod} \\ m &= \textit{mass of the bobs} \\ g &= \textit{gravitational constant} \end{aligned}$$

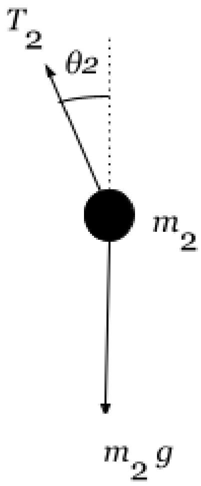
Figure 2: Forces acting on the first pendulum

Then, according to the forces drawn above by referring to fig. 1. Newton's equation read the following two:

$$m_1 x_1'' = -T_1 \sin(\theta_1) + T_2 \sin(\theta_2) \tag{13}$$

$$m_1 y_1'' = T_1 \cos(\theta_2) - T_2 \cos(\theta_2) - m_1 g \tag{14}$$

On the other hand, for the lower pendulum, there is a slight change that needs to be taken into account.



$$m_2 x_2'' = -T_2 \sin(\theta_2) \tag{15}$$

$$m_2 y_2'' = T_2 \cos(\theta_2) - m_2 g \tag{16}$$

Figure 3: Forces acting on the second pendulum

3.1.3. Defining the equation of motion

Next, let's do some algebraic manipulations with the objective to find expressions for (θ_1'', θ_2'') in terms of $(\theta_1, \theta_1', \theta_2, \theta_2')$. Let's start by solving the equations (15), and (16) for $T_2 \sin(\theta_2)$, and $T_2 \cos(\theta_2)$ and subsequently, substitute in equations (13), and (14).

$$m_1 x_1'' = -T_1 \sin(\theta_1) - m_2 x_2'' \tag{17}$$

$$m_1 y_1'' = T_1 \cos(\theta_1) - m_2 y_2'' - m_2 g - m_1 g \tag{18}$$

If we multiply both equations above by $\sin(\theta_1)$ we will get the following equations:

$$T_1 \sin(\theta_1)\cos(\theta_1) = -\cos(\theta_1)\left(m_1 x_1'' + m_2 x_2''\right) \tag{19}$$

$$T_1 \sin(\theta_1)\cos(\theta_1) = \sin(\theta_1)\left(m_1 y_1'' + m_2 y_2'' + m_2 g + m_1 g\right) \tag{20}$$

This results in the following equation:

$$\sin(\theta_1)\left(m_1 y_1'' + m_2 y_2'' + m_2 g + m_1 g\right) \ = -\cos(\theta_1)\left(m_1 x_1'' + m_2 x_2''\right) \tag{21}$$

Next, let's multiply equations (15) and (16) by $\cos(\theta_2)$ and $\sin(\theta_2)$ respectively and reorganize to get:

$$T_2 \sin(\theta_2)\cos(\theta_2) = -\cos(\theta_2)\left(m_2 x_2''\right) \tag{22}$$

$$T_2 \sin(\theta_2)\cos(\theta_2) = \sin(\theta_2)\left(m_2 y_2'' + m_2 g\right) \tag{23}$$

Equating the right-hand side gives:

$$\sin(\theta_2)\left(m_2 y_2'' + m_2 g\right) = -\cos(\theta_2)\left(m_2 x_2''\right) \tag{24}$$

In order to continue solving for θ_1'', θ_2'', we would need a computer algebra program to be able to solve the equations (21), and (24).

$$\theta_1'' = \frac{-g(2m_1 + m_2)\sin(\theta_1) - m_2 g \sin(\theta_1 - 2\theta_2) - 2\sin(\theta_1 - \theta_2) m_2 \left((\theta_2')^2 l_2 + (\theta_1')^2 l_1 \cos(\theta_1 - \theta_2)\right)}{L_1(2m_1 + m_2 - m_2\cos(2\theta_1 - 2\theta_2))} \tag{25}$$

$$\theta_2'' = \frac{2\sin(\theta_1 - \theta_2)\left((\theta_1')^2 l_1 (m_1 + m_2) + g(m_1 + m_2)\cos(\theta_1) + (\theta_2')^2 l_2 m_2 \cos(\theta_1 - \theta_2)\right)}{L_1(2m_1 + m_2 - m_2\cos(2\theta_1 - 2\theta_2))} \tag{26}$$

Therefore, after all these algebraic calculations we were able to derive the Newtonian equation of motion regarding the double pendulum system.

3.2. Equations of motion from Euler-Lagrange

In Lagrangian mechanics, the dynamics of a system are described in terms of generalised coordinates and generalised velocities. In our context, the generalised variables are the pendulum deflection angles (θ_1, θ_2) and the angular velocities (θ'_1, θ'_2). Using these parameters, we generate the lagrangian for the double pendulum and formulate the Euler-Lagrange differential equations. In this section (3.2.), we assume that both rods are massless and that all pivots are assumed to be frictionless.

To begin, the Lagrangian for this particular system will also consist of the equations (1) to (4) which are the positions (x_1, y_1, x_2, y_2) of the double pendulum.

The Lagrangian for any system is given by $L = T - V$, where T and V are the kinetic and potential energies of the system respectively.

The potential energy (V) is given by the following:

$$V = m_1 g y_1 + m_2 g y_2$$

$$V = -m_1 g l_1 \cos(\theta_1) - m_2 g l_2 (l_1 \cos\theta_1 + l_2 \cos\theta_2)$$

$$V = -(m_1 + m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2 \qquad (27)$$

The kinetic energy (T) is given by the following:

$$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

$$T = \frac{1}{2} m_1 \left((x'_1)^2 + (y'_1)^2 \right) + \frac{1}{2} m_2 \left((x'_2)^2 + (y'_2)^2 \right)$$

$$T = \frac{1}{2} m_1 \left(l_1^2 (\theta'_1)^2 \cos^2\theta_1 + l_1^2 (\theta'_1)^2 \sin^2\theta_1 \right)$$

$$+ \frac{1}{2} m_2 \left[\left(l_1 \theta'_1 \cos\theta_1 + l_2 \theta'_2 \cos\theta_2 \right)^2 + \left(l_1 \theta'_1 \sin\theta_1 + l_2 \theta'_2 \sin\theta_2 \right)^2 \right]$$

$$T = \frac{1}{2} m_1 l_1^2 (\theta'_1)^2 + \frac{1}{2} m_2 \left[l_1^2 (\theta'_1)^2 \cos^2\theta_1 + l_2^2 (\theta'_2)^2 \cos^2\theta_2 \right]$$

$$+ \frac{1}{2} m_2 \left[2 l_1 l_2 \theta'_1 \theta'_2 \cos\theta_1 \cos\theta_2 + l_1^2 (\theta'_1)^2 \sin^2\theta_1 + l_2^2 (\theta'_2)^2 \sin^2\theta_2 + 2 l_1 l_2 \theta'_1 \theta'_2 \sin\theta_1 \sin\theta_2 \right]$$

$$T = \frac{1}{2} m_1 l_1^2 (\theta'_1)^2 + \frac{1}{2} m_2 \left[l_1^2 (\theta'_1)^2 + l_2^2 (\theta'_2)^2 + 2 l_1 l_2 \theta'_1 \theta'_2 \cos(\theta_1 - \theta_2) \right] \qquad (28)$$

After determining both kinetic and potential energy, the Lagrangian of the system can be determined:

$$L = T - V$$

$$L = \frac{1}{2}(m_1 + m_2)l_1^2(\theta_1')^2 + \frac{1}{2}m_2 l_2^2(\theta_2')^2 + m_2 l_1 l_2 \theta_1' \theta_2' \cos(\theta_1 - \theta_2)$$
$$+ (m_1 + m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2 \qquad (29)$$

Now, we can derive the Euler Lagrange equations.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \theta_i'}\right) - \frac{\partial L}{\partial \theta_i} = 0 \qquad (30)$$

Lets us start with θ_1:

$$\frac{\partial L}{\partial \theta_1'} = (m_1 + m_2) l_1^2 \theta_1' + m_2 l_1 l_2 \theta_2' \cos(\theta_1 - \theta_2) \qquad (31)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \theta_1'}\right) = (m_1 + m_2) l_1^2 \theta_1'' + m_2 l_1 l_2 \theta_2'' \cos(\theta_1 - \theta_2)$$
$$- m_2 l_1 l_2 \theta_2' \sin(\theta_1 - \theta_2)(\theta_1' - \theta_2') \qquad (32)$$

$$\therefore, \frac{\partial L}{\partial \theta_1} = -l_1 g(m_1 + m_2)\sin(\theta_1) - m_2 l_1 l_2 \theta_1' \theta_2' \sin(\theta_1 - \theta_2) \qquad (33)$$

So, the first Euler-Lagrange differential equations for θ_1 becomes the following (after dividing by l_1):

$$(m_1 + m_2) l_1 \theta_1'' + m_2 l_2 \theta_2'' \cos(\theta_1 - \theta_2) + m_2 l_2 (\theta_2')^2 \sin(\theta_1 - \theta_2) + g(m_1 + m_2)\sin\theta_1 = 0 \qquad (34)$$

Similarly for θ_2:

$$\frac{\partial L}{\partial \theta_2'} = m_2 l_2^2 \theta_2' + m_2 l_1 l_2 \theta_1' \cos(\theta_1 - \theta_2) \tag{35}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \theta_2'}\right) = m_2 l_2^2 \theta_2'' + m_2 l_1 l_2 \theta_1'' \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \theta_1' sin(\theta_1 - \theta_2)\left(\theta_1' - \theta_2'\right) \tag{36}$$

$$\therefore, \frac{\partial L}{\partial \theta_2} = m_2 l_1 l_2 \theta_1' \theta_2' \sin(\theta_1 - \theta_2) - l_2 m_2 g \sin\theta_2 \tag{37}$$

So, the second Euler-Lagrange differential equations for θ_2 becomes the following (after dividing by $m_2 l_2$):

$$l_2\theta_2'' + l_1\theta_1'' cos(\theta_1 - \theta_2) - l_1\left(\theta_1'\right)^2 \sin(\theta_1 - \theta_2) + g\sin\theta_2 = 0 \tag{38}$$

This pair of equations (34) and (38) is equivalent to the Newtonian equations (25) and (26). The Lagrangian provides a much more direct and easier way of solving complicated systems such as this one. Unlike the Newtonian way where one must account for constraints which are the conditions needed to keep the object in question confined to a space.

3.3. Derivation of the Hamilton Equations using Legendre Transformation

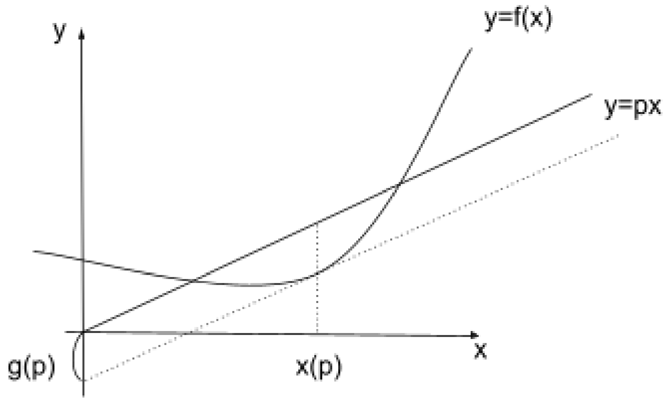
Looking back to the initial nonlinear system of equations, we now investigate the behaviour of oscillations of arbitrary amplitude. This equation system cannot be solved analytically. We will look at a numerical model of the double pendulum. The above derived Lagrange equations are second-order differential equations. Converting them into the form of Hamilton's canonical equations is more convenient. As a consequence, instead of the two second-order equations, we get a system of four first-order differential equations. Moreover, in Hamiltonian mechanics, the state of a system is given by the generalised coordinates and generalised momenta. In our situation, we can employ the angles (θ_1, θ_2) as generalised coordinates, as we did in the Lagrange equations. Rather than generalised velocities (θ_1', θ_2') (as in the Lagrangian), we now add generalised momenta (p_1, p_2) that correspond to the velocities through the following formulas.

$$p_1 = \frac{\partial L}{\partial \theta_1'}, \; and \; p_2 = \frac{\partial L}{\partial \theta_2'} \tag{39}$$

In order to transit from the Lagrangian to the Hamiltonian form of equations is achieved by using the Legendre transformation. It is defined as follows.

Suppose that f(x) is a smooth convex downward function which can be visualised as in figure 4.

Figure4: convex downward function



Here the line y=px passes through the origin and the distance in between this line and the convex function y=f(x) along the y-axis depends on the coordinate x. That distance will be greatest at a certain value of y. In essence, it is determined by the slope of the line that is on a parameter p. Hence, we introduce a new function g(p):

$$g(p) = \max_{x}(px - f(x)) \qquad (40)$$

The Legendre transformation refers to the transformation of the function f(x) into its conjugate function g(p). It is worth noting that the function g(p) achieves its maximum value with regard to the variable x when p=(df/dx)

$$\frac{d}{dx}(px - f(x)) = p - \frac{df(x)}{dx} = 0, \ \Rightarrow p = \frac{df(x)}{dx} = p(x) \qquad (41)$$

We can determine the inverse function x(p) by knowing the dependency p(x). The Legendre transform is then expressed by the following relationship:

$$g(p) = px(p) - f(x(p)), \, where \, p = \frac{df}{dx} \qquad (42)$$

The Legendre transformation is easily adapted to functions with several variables. The Legendre transformation of the form below describes the transition from the Lagrangian to the Hamiltonian in the double pendulum model.

$$H(\theta_1, \theta_2, p_1, p_2) = \sum_{i=1}^{2} \theta'_i p_i - L(\theta_1, \theta_2, \theta'_1, \theta'_2) = \theta'_1 p_1 + \theta'_2 p_2 - L(\theta_1, \theta_2, \theta'_1, \theta'_2,) \quad (43)$$

$$where,\ p_1 = \frac{\partial L}{\partial \theta'_1}\ and,\ p_2 = \frac{\partial L}{\partial \theta'_2}$$

In the previous statement, L is the Lagrangian, and the function H is the system's Hamiltonian, which is dependent on the generalised coordinates θ_1, θ_2 and generalised momenta p_1, p_2.

Each Lagrange equation is transformed into a system of two Hamilton's canonical equations of the following form:

$$\frac{d}{dt}\frac{\partial L}{\partial \theta'_i} = \frac{\partial L}{\partial \theta_i}, \quad \Rightarrow p'_i = -\frac{\partial H}{\partial \theta_i} \quad (44)$$

$$\Rightarrow \theta'_i = \frac{\partial H}{\partial p_i}$$

We express the Hamilton's canonical equations for the double pendulum in the final form, without presenting the explicit algebraic transformations of the Laplace transform.

$$\theta'_1 = \frac{p_1 l_2 - p_2 l_1 \cos(\theta_1 - \theta_2)}{l_1^2 l_2 \left[m_1 + m_2 \sin^2(\theta_1 - \theta_2)\right]} \quad (45)$$

$$\theta'_1 = \frac{p_2(m_1 + m_2) l_1 - p_1 m_2 l_2 \cos(\theta_1 - \theta_2)}{m_2 l_1 l_2^2 \left[m_1 + m_2 \sin^2(\theta_1 - \theta_2)\right]} \quad (46)$$

$$p'_1 = -g l_1 (m_1 + m_2) \sin(\theta_1) - A_1 + A_2 \quad (47)$$

$$p'_2 = -m_2 g l_2 \sin(\theta_2) + A_1 - A_2 \quad (48)$$

Where

$$A_1 = \frac{p_1 p_2 \sin(\theta_1 - \theta_2)}{l_1 l_2 \left[m_1 + m_2 \sin^2(\theta_1 - \theta_2)\right]} \quad (49)$$

$$A_2 = \frac{1}{2 l_1^2 l_2^2 \left[m_1 + m_2 \sin^2(\theta_1 - \theta_2)\right]^2} \left[p_1^2 m_2 l_2^2 - 2 p_1 p_2 m_2 l_1 l_2 \cos(\theta_1 - \theta_2) + p_2^2 (m_1 + m_2) l_1^2\right] \sin\left[2(\theta_1 - \theta_2)\right] \quad (50)$$

Legendre transformation was used here to go from the Lagrangian to the Hamilton formalism. In essence, it converts a function of one set of variables to another function of a conjugate set of variables. They both represent the same system except that the Lagrangian system is a subset of the Hamiltonian system.

3.4. Numerical Simulation of Chaotic Oscillations

The 4/5 pair order Runge-Kutta technique is the most often used method for numerically solving differential equations. Most mathematical programs employ various variants of this strategy often with automatic error control and adaptive time-stepping. This includes the Python code that was written in order to simulate a double pendulum system. The "solve_ivp" command from the Scipy package utilises this algorithm for its computations

In order to simulate the motion of a double pendulum, we first employ the classical 4th order Runge-Kutta technique. We simplify the differential equations by assuming that the pendulum lengths are the same ($l_1 = l_2 = l$). By introducing the mass ratio-related parameter μ, it gives the ratio of both masses where:

$$\mu = \frac{m_2}{m_1} \tag{51}$$

The system may be written as follows:

$$\theta_1' = \frac{p_1 - p_2 \cos(\theta_1 - \theta_2)}{m_1 l^2 \left[1 + \mu \sin^2(\theta_1 - \theta_2)\right]} \tag{52}$$

$$\theta_1' = \frac{p_2(1+\mu) - p_1 \mu \cos(\theta_1 - \theta_2)}{m_1 l^2 \left[1 + \mu \sin^2(\theta_1 - \theta_2)\right]} \tag{53}$$

$$p_1' = -gl(1+\mu)\sin(\theta_1) - A_1 + A_2 \tag{54}$$

$$p_2' = -m_1 \mu g l \sin(\theta_2) + A_1 - A_2 \tag{55}$$

Where,

$$A_1 = \frac{p_1 p_2 \sin(\theta_1 - \theta_2)}{m_1 l^2 \left[1 + \mu \sin^2(\theta_1 - \theta_2)\right]} \tag{56}$$

$$A_2 = \frac{1}{2 m_1 l^2 \left[1 + \mu \sin^2(\theta_1 - \theta_2)\right]^2} \left[p_1^2 \mu - 2 p_1 p_2 \mu \cos(\theta_1 - \theta_2) + p_2^2 (1+\mu)\right] \sin\left[2(\theta_1 - \theta_2)\right] \tag{57}$$

This system may be rewritten in vector form:

$$Z' = f(Z),\ where\ Z = (\theta_1,\ \theta_2,\ p_1,\ p_2)^T,\ f = (f_1,\ f_2,\ f_3,\ f_4)^T \tag{58}$$

The vector Z is made up of 4 canonical system variables, and its components of the vector correspond to the right-hand sides of the differential equations.
The Runge-Kutta approach necessitates the sequential assessment of the four intermediate vectors at each step where they can be seen below:

$$Y_1 = \tau f(Z_n),\ Y_2 = \tau f\left(Z_n + \frac{1}{2}Y_1\right),\ Y_3 = \tau f\left(Z_n + \frac{1}{2}Y_2\right),\ Y_4 = \tau f(Z_n + Y_3) \qquad (59)$$

The vector's value in the following node is determined by the time step forward:

$$Z_{n+1} = Z_n + \frac{1}{6}(Y_1 + 2Y_2 + 2Y_3 + Y_4) \qquad (60)$$

The same problem is then solved using a more accurate 5th order Runge-Kutta algorithm, which makes use of a vector composed of 5 variables. If the error between the two solutions is significant enough, then the step size is reduced. The total error of the procedure on a finite interval has the order $O(\tau^4)$, which means that the computational accuracy grows by 16 times while the time step τ is reduced by twice. This allows the algorithm to adapt to every application, making it simple to not only determine the accuracy of the results, but also increase it if necessary.
The specified model is depicted in the animation. For the sake of simplicity, we assume that the pendulum's starting deflection angles are equal. This application shows the chaotic dynamics of the double pendulum for various values of μ and θ. The nonlinear dynamics of a double pendulum appear to be understudied by physicists and mathematicians and hold many surprises.

3.5. Small oscillations of the Double Pendulum

The oscillations of the pendulums at the zero equilibrium point may be explained by a linear set of equations if the angles are minimal. To obtain such a system, let us return to the system's initial Lagrangian which refers back to equation 29. However, let us rewrite the Lagrangian in a simpler form, expanding it in a Maclaurin series and retaining the linear and quadratic terms. The trigonometric functions can be approximated using the following expressions:

$$\cos\theta_1 \approx 1 - \frac{\theta_1^2}{2},\ \ \cos\theta_2 \approx 1 - \frac{\theta_2^2}{2},\ \ \cos(\theta_1 - \theta_2) \approx 1 - \frac{(\theta_1 - \theta_2)^2}{2} \approx 1$$

We took into account the fact that the latter expression comprises the product of tiny numbers (θ_1', θ_2') and has the second order of smallness. As a result, we can only leave the linear term in the cosine expansion. By substituting this into the Lagrangian and assuming that the potential energy is specified up to an integer, we acquire the quadratic Lagrangian for the double pendulum as follows:

$$L = T - V = \frac{1}{2}(m_1 + m_2)l_1^2(\theta_1')^2 + \frac{1}{2}m_2 l_2^2(\theta_2')^2 + m_2 l_1 l_2 \theta_1' \theta_2' - \frac{1}{2}(m_1 + m_2) g l_1 \theta_1^2 + \frac{1}{2} g l_2 \theta_2^2 \qquad (61)$$

Now, we derive the Lagrange differential equation for the above Lagrangian which are written as follows:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \theta_1'}\right) - \frac{\partial L}{\partial \theta_1} = 0\,, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \theta_2'}\right) - \frac{\partial L}{\partial \theta_2} = 0 \tag{62}$$

The partial derivatives can be found:

$$\frac{\partial L}{\partial \theta_1'} = (m_1 + m_2) l_1^2 \theta_1' + m_2 l_1 l_2 \theta_2' \tag{63}$$

$$\frac{\partial L}{\partial \theta_1} = -(m_1 + m_2) g l_1 \theta_1 \tag{64}$$

$$\frac{\partial L}{\partial \theta_2'} = m_2 l_2^2 \theta_2' + m_2 l_1 l_2 \theta_1' \tag{65}$$

$$\frac{\partial L}{\partial \theta_2} = -m_2 g l_2 \theta_2 \tag{66}$$

We end up with a system of two differential equations:

$$\frac{d}{dt}\left[(m_1 + m_2) l_1^2 \theta_1' + m_2 l_1 l_2 \theta_2'\right] + (m_1 + m_2) g l_1 \theta_1 = 0 \tag{67}$$

$$\frac{d}{dt}\left[m_2 l_2^2 \theta_2' + m_2 l_1 l_2 \theta_1'\right] + m_2 g l_2 \theta_2 = 0 \tag{68}$$

Or

$$(m_1 + m_2) l_1^2 \theta_1'' + m_2 l_1 l_2 \theta_2'' + (m_1 + m_2) g l_1 \theta_1 = 0 \tag{69}$$

$$m_2 l_2^2 \theta_2'' + m_2 l_1 l_2 \theta_1'' + m_2 g l_2 \theta_2 = 0 \tag{70}$$

This system of equations can be expressed in the form of a compact matrix below:

$$\theta(t) = \begin{bmatrix} \theta_1(t) \\ \theta_2(t) \end{bmatrix},\ M = \begin{bmatrix} (m_1 + m_2) l_1^2 & m_2 l_1 l_2 \\ m_2 l_1 l_2 & m_2 l_2^2 \end{bmatrix},\ and\, K = \begin{bmatrix} (m_1 + m_2) g l_1 & 0 \\ 0 & m_2 g l_2 \end{bmatrix}$$

The differential equation system may then be expressed as

$$M\theta'' + K\theta = 0 \tag{71}$$

This equation represents the free undamped oscillations with a given frequency in the situation of a single body. In the scenario of the double pendulum, the solution (as shown below) will contain oscillations with two distinct frequencies, known as normal modes. The complex-valued vector function's normal modes are its real component.

$$\theta(t) = \begin{bmatrix}\theta_1(t)\\ \theta_2(t)\end{bmatrix} = Re\left(\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix} e^{iwt}\right) \tag{72}$$

Where ξ_1, ξ_2 are the eigenvectors, ω is the real frequency. The normal frequencies $\omega_{1,2}$ values are found by solving the following auxiliary equation.

$$\det\left(K - \omega^2 M\right) = 0 \tag{73}$$

When it comes to arbitrary masses m_1, m_2, as well as lengths l_1, l_2 the auxiliary equation looks like this:

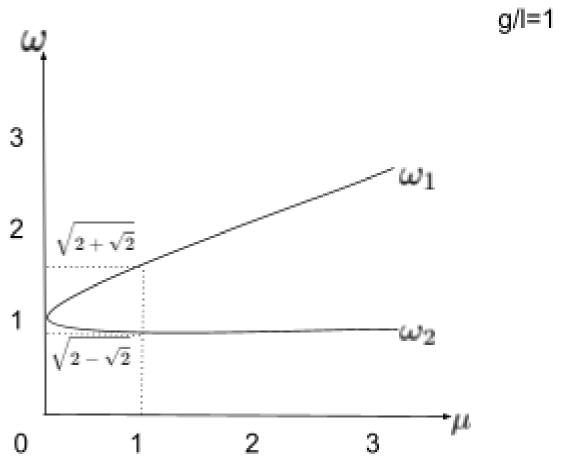
$$(m_1 + m_2)g^2 - \omega^2(m_1 + m_2)(l_1 + l_2)g + \omega^4 m_1 l_1 l_2 = 0 \tag{74}$$

Hence, we get a biquadratic equation for frequencies ω. The general solution to this equation is a little complicated. Therefore, we investigate the scenario in which the lengths of both pendulums' rods are equal. A concise formula will then be used to determine the normal frequencies.

$$\omega_{1,2}^2 = \frac{g}{l}\left[1 + \mu \pm \sqrt{(1+\mu)\mu}\right],\ where\, \mu = \frac{m_2}{m_1} \tag{75}$$

As seen above, the eigenfrequencies $\omega_{1,2}$ are solely determined by the mass ratio μ. The frequencies dependence ω_1, ω_2 based on the parameter μ (given $\frac{g}{l} = 1$) are depicted in Figure 5 below.

Figure 5: the frequencies dependency on μ



When both masses are equivalent, which occurs when μ is equal to 1, the frequencies are given by:

$$w_{1,\,2} = \sqrt{\frac{g}{l}}\sqrt{2 \pm \sqrt{2}} \qquad (76)$$

After determining the eigenfrequencies $\omega_{1,\,2}$, we must now find the eigenvectors $\xi_1,\ \xi_2$ to characterize the normal modes. Solving the vector-matrix equation yields them.

$$\left(K - \omega^2 M\right)\xi = 0 \qquad (77)$$

Assume that the eigenvector $\xi_1 = \left(\xi_{11},\ \xi_{21}\right)^T$ corresponds to the normal frequency ω_1. Then, we obtain the following expression for ξ_1:

$$\left(K - \omega_1^2 M\right)\xi_1 = 0 \Rightarrow m_1 l \begin{bmatrix} (1+\mu)\left(g - \omega_1^2 l\right) & -\omega_1^2 \mu l \\ -\omega_1^2 \mu l & \mu\left(g - \omega_1^2 l\right) \end{bmatrix} \begin{bmatrix} \xi_{11} \\ \xi_{21} \end{bmatrix} = 0 \qquad (78)$$

Where ω_1 is equal to equation (75)

The eigenvectors ξ_1 coordinates satisfy the equation.

$$(1+\mu)\left(g - \omega_1^2 l\right)\xi_{11} - \omega_1^2 \mu l \xi_{21} = 0 \qquad (79)$$

$$\Rightarrow \frac{\xi_{21}}{\xi_{11}} = -\frac{(1+\mu)\left[\mu + \sqrt{(1+\mu)\mu}\right]}{\mu\left[1 + \mu + \sqrt{(1+\mu)\mu}\right]} = -\sqrt{\frac{1+\mu}{\mu}} \qquad (80)$$

Therefore, the eigenvector ξ_1 is determined by:

$$\xi_1 = \begin{bmatrix} \xi_{11} \\ \xi_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ -\sqrt{\frac{1+\mu}{\mu}} \end{bmatrix} \tag{81}$$

Correspondingly, the coordinates of the second eigenvector are determined by
$\xi_2 = (\xi_{12},\, \xi_{22})^T$
It is found to be:

$$\xi_2 = \begin{bmatrix} \xi_{12} \\ \xi_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{\frac{1+\mu}{\mu}} \end{bmatrix} \tag{82}$$

At last, the matrix equation's general solution may be stated as follows:

$$\theta(t) = \begin{bmatrix} \theta_1(t) \\ \theta_2(t) \end{bmatrix} = Re\left(\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} e^{i\omega t} \right) = C_1 \begin{bmatrix} 1 \\ -\sqrt{\frac{1+\mu}{\mu}} \end{bmatrix} \cos\left(\omega_1 t + \phi_1\right)$$

$$+ C_2 \begin{bmatrix} 1 \\ \sqrt{\frac{1+\mu}{\mu}} \end{bmatrix} \cos\left(\omega_2 t + \phi_2\right) \tag{83}$$

Where, $C_1,\, C_2,\, \phi_1,\, \phi_2$ depends on the pendulum's initial positions and velocities.

Let us consider the behaviour of tiny oscillations for a certain set of initial data. Assume, for instance, that the pendulums' initial positions and velocities are as follows:

$$\theta_1(t=0) = 0,\, \theta_2(t=0) = \frac{\pi}{6},\, \theta_1'(t=0) = 0,\, \theta_2'(t=0) = 0$$

In this particular instance, both initial phases are equal to 0. Thus, we would only have to solve for both $C_1,\, C_2$.

$$\theta_1(0) = C_1 + C_2 = 0$$

$$\theta_2(0) = -C_1\sqrt{\frac{1+\mu}{\mu}} + C_2\sqrt{\frac{1+\mu}{\mu}} = \frac{\pi}{6}$$

$$\Rightarrow C_1 = -C_2 \Rightarrow 2C_2\sqrt{\frac{1+\mu}{\mu}} = \frac{\pi}{6}$$

$$C_2 = \frac{\pi}{12}\sqrt{\frac{\mu}{1+\mu}},\, and\, C_1 = -\frac{\pi}{12}\sqrt{\frac{\mu}{1+\mu}}$$

Now that we obtained both constants, we can apply the law of oscillations of the pendulum.

$$\theta_1(t) = -\frac{\pi}{12}\sqrt{\frac{\mu}{1+\mu}}\cos(\omega_1 t) + \frac{\pi}{12}\sqrt{\frac{\mu}{1+\mu}}\cos(\omega_2 t)$$

$$\theta_2(t) = \frac{\pi}{12}\cos(\omega_1 t) + \frac{\pi}{12}\cos(\omega_2 t)$$

And the angular frequencies ω_1, ω_2 are expressed by the square root of the equation 75.

$$\omega_{1,2} = \sqrt{\frac{g}{l}}\sqrt{1+\mu \pm \sqrt{(1+\mu)\mu}}.$$

It is important to notice that in the following figures 6 - 8, the angles $\theta_1(t)$ and $\theta_2(t)$ are in radians and the time t in seconds. They demonstrate the small oscillations for three distinct values of μ. The first being equal to 0.2, the second being equal to 1, and the last being equal to 5. Both lengths of the pendulums are equal to 0.25m, and the gravitational constant is equal to 9.8 m/s^2. For simplicity, the deflection angles of the pendulums are provided in degrees.

Figure 6: $\mu = 0.2$

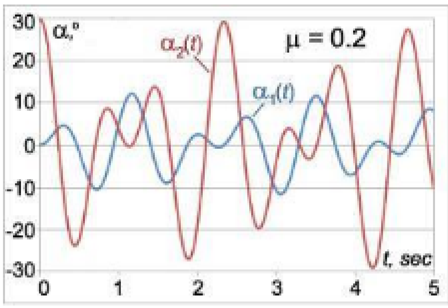


Figure 7: $\mu = 1$

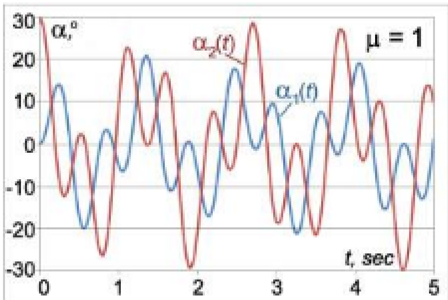
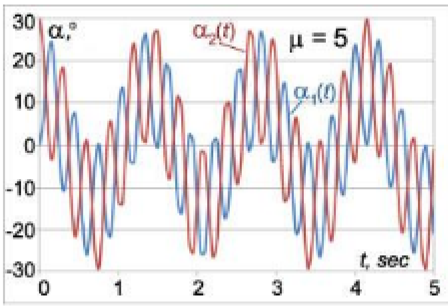


Figure 8: $\mu = 5$



After analysing the above graphs, we can see that the energy is being transmitted cyclically between the two masses. When one of the pendulums comes to a halt, the other swings with the maximum amplitude. The curves "swap roles" after a while, and so forth.

The resulting oscillations are depicted as higher frequency ($\omega = \frac{\omega_1 + \omega_2}{2}$) oscillations with a frequency-dependent periodic amplitude modulation ($\Delta\omega = \frac{\omega_1 - \omega_2}{2}$) that are referred to as the beat frequency.

Thus, the little oscillations of the double pendulum appear to be periodic variations and are characterised by the sum of two harmonics with frequencies ω_1, ω_2 based on the factors of the system

The small oscillations of the double pendulum will be periodic if the ratio of the eigenfrequencies ω_1, ω_2 is equal to a rational integer. However, if the frequency ratio is an irrational value, the small oscillations cannot be periodic.

4. Python simulation

In order to observe the movement of the system, a simulation that depicts it in detail must be constructed. This was done using the Python programming language due to its ease of use and its open source nature.
First, the initial conditions were set as follows:

Table 1: The parameters and initial conditions of the first simulation

g	9.8
L_1	2
L_2	1.5
m_1	1
m_2	1
θ_1	$\frac{\pi}{4}$
θ_1'	1
θ_2	$\frac{\pi}{3}$
θ_2'	2

These values in conjunction with equations (25) and (26) were then used to set up a system of first-order differential equations:

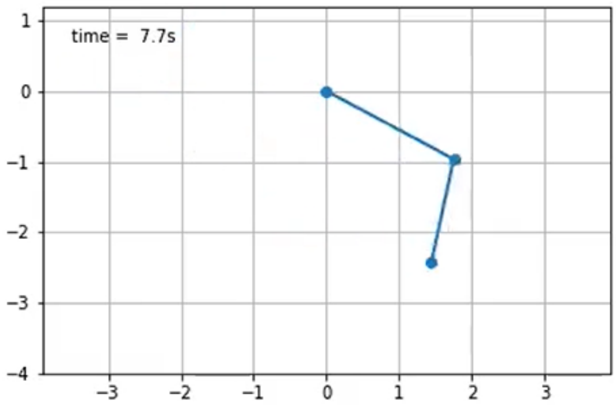
$$u_1' = u_3$$
$$u_2' = u_4$$
$$u_3' = \frac{-g(2m_1+m_2)\sin(u_1) - m_2 g\sin(u_1-2u_2) - 2\sin(u_1-u_2)m_2\left({u_4}^2L_2+{u_3}^2L_1\cos(u_1-u_2)\right)}{L_1(2m_1+m_2-m_2\cos(2u_1-2u_2))}$$

$$u_4' = \frac{2\sin(u_1-u_2)\left({u_3}^2L_1(m_1+m_2)+g(m_1+m_2)\cos(u_1)+{u_4}^2L_2m_2\cos(u_1-u_2)\right)}{L_2(2m_1+m_2-m_2\cos(2u_1-2u_2))}$$

where $u_1 = \theta_1,\ u_2 = \theta_2,\ u_3 = \theta_1',\ u_4 = \theta_2'$.

This system was then computed numerically using the Runge-Kutta 4/5 algorithm. Despite it being an approximation rather than an analytical solution, this algorithm is still adequate for the purposes of this simulation. The resulting values were then converted to cartesian coordinates using equations (1) to (4). Each of these sets of coordinates represent the locations of the point masses at the end of the pendulums and correspond to a specific point in time. In order to animate the system, these frames are put in a sequence so that they represent a smooth motion of the pendulums. For each frame, a line is drawn from the anchor point (0,0) to the mass of the first pendulum, then to the mass of the second pendulum.

Figure 9: An example of an animation frame



After this first simulation was computed, two more are included. For each of these, one of the values within the initial conditions was altered by a minuscule amount (10^{-6}). The altered variables are the initial and the initial position of the first pendulum.

The code can be found in its entirety at the following web address: https://github.com/Vlad-Andrei-Stelea/Double-Pendulum/blob/main/code.

5. Results

The simulations were run for 25 seconds each. Using these results, the following graphs have been constructed:

Figure 10: The trajectory of the second pendulum's mass during the original simulation

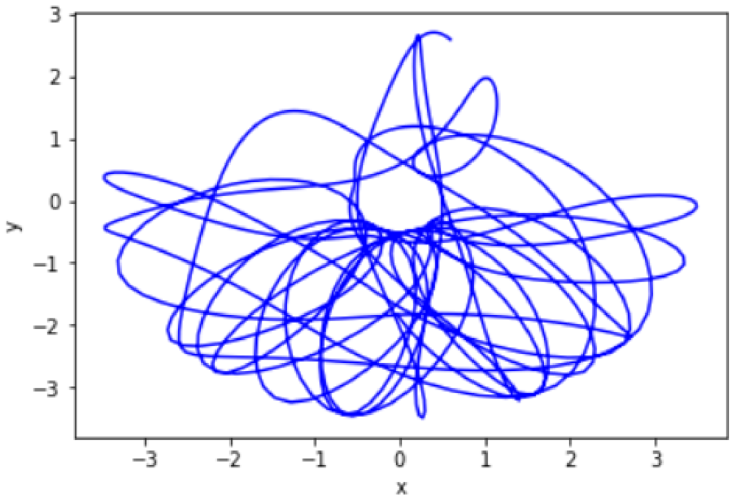


Figure 11: The x coordinates of the second pendulum's mass in the first (blue), second (green) and third (red) simulations over time

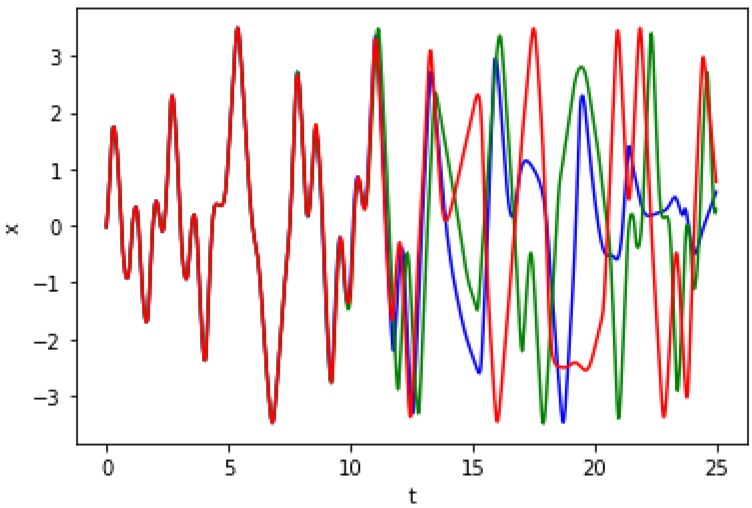
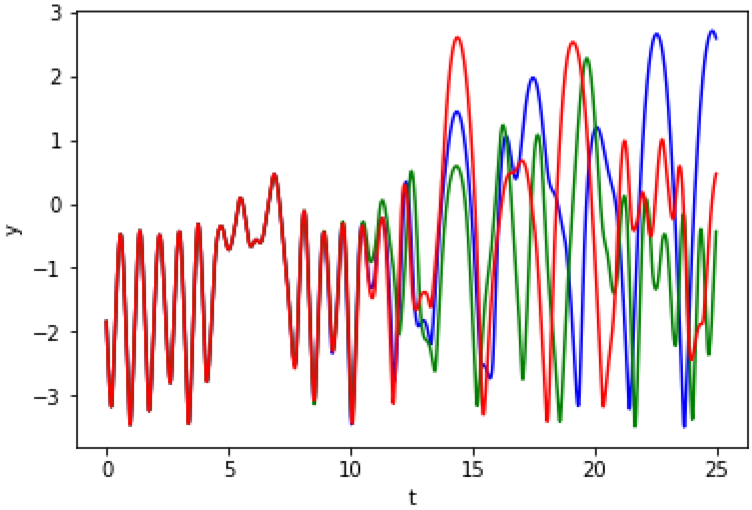


Figure 12: The y coordinates of the second pendulum's mass in the first (blue), second (green) and third (red) simulations over time



6. Conclusion

First, we have laid out and analysed the theory behind the motion of a double pendulum system. Then using this knowledge, a simulation showcasing this was created in order to study the system's motion and its chaotic movement.
It can be observed that there are major differences between these simulations, despite their initial conditions only varying by a minuscule amount. At first their behaviour is identical, but after a short period of time, they deviate to a significant degree. This is proof of the chaotic behaviour of these pendulums, as any change in the initial conditions, however small, will lead to a drastically different movement.
However, one fact to keep in mind is that these simulations were done using the Runge-Kutta 45 algorithm, which, despite its good accuracy, is not totally accurate. Small errors are introduced with each frame of animation, and they accumulate to a non-insignificant amount. Therefore, while these animations are adequate at illustrating the movements of the pendulums, they grow more inaccurate with each second.
However, it must be mentioned that these simulations were calculated under perfect conditions. In the real world, the pendulums have their mass distributed throughout their length, and they are also affected by both the friction within their joints and the air drag.

7. References

"Double Pendulum." *MyPhysicsLab*,
https://www.myphysicslab.com/pendulum/double-pendulum-en.html

Weisstein, Eric. "Double Pendulum." *WOLFRAM RESEARCH*,
https://scienceworld.wolfram.com/physics/DoublePendulum.html

Svirin, Alex. "Double Pendulum." *Math24*, https://math24.net/double-pendulum.html

Assencio, Diego. "The Double Pendulum: Lagrangian Formulation." *Diego.assencio.com*,
https://diego.assencio.com/?index=1500c66ae7ab27bb0106467c68feebc6

Mathews et al. "Numerical Methods Using Matlab 4th Edition". P. 497 to 499.
https://maths.cnam.fr/IMG/pdf/RungeKuttaFehlbergProof.pdf