3.5 Implicit Differentiation

The functions we have been differentiating so far have been explicitly expressing one variable in terms of another.

ex: $y = \sqrt{x^3 + x + 1}$ or, more generally, $y = f(x)$

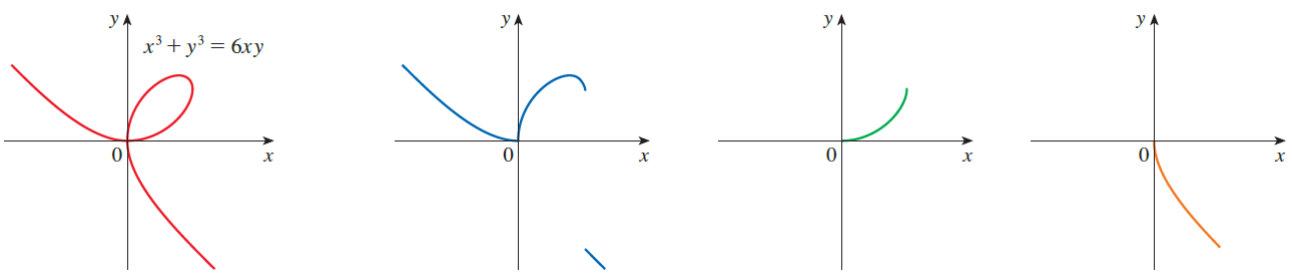
But some functions are defined implicitly by a relation between x and y

ex: The circle $x^2 + y^2 = 25$ or the folium of Descartes $x^3 + y^3 = 6xy$

Sometimes we can solve such an equation y as 1 or more explicit functions

> **Example 1**
>
> Solve the equation of the circle $x^2 + y^2 = 25$ explicitly for y

For other equations, it's either not easy or even possible to solve them explicitly. Ex: $x^3 + y^3 = 6xy$



Fortunately, we don't need to make the equations explicit in order to find y'

Result 1: Implicit Differentiation

Differentiate both sides of the equation with respect to x, then solve for $\frac{dy}{dx} = y'$.

Remember, we treat y as a function of x.

$\frac{d}{dx}x = 1$ and $\frac{d}{dy}y = 1$

whereas $\frac{d}{dx}y = \frac{d}{dx}f(x) = f'(x) = y'$

Example 2

1. If $x^2 + y^2 = 25$, find $\frac{dy}{dx}$
2. Find an equation of the tangent line to the circle at the point (3,4)

Example 3

1. Find y' for $x^3 + y^3 = 6xy$
2. Find an equation of the tangent line to the circle at the point (3,3)
3. At what point with $x > 0$ is the tangent line horizontal?

Example 4

Find y' for $\sin(x+y) = y^2 \cos x$

Example 5

Find y'' for $x^4 + y^4 = 16$

Derivatives of Inverse Trig Functions

We can use implicit differentiation to find the derivatives of inverse trig functions without using the limit definition.

Example 6

Find the derivative of $y = \sin^{-1} x = \arcsin x$

Example 7

Find the derivative of $y = \tan^{-1} x = \arctan x$

Result 2: Derivatives of Trigonometric Functions

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2} \qquad \frac{d}{dx}\cot^{-1} x = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}\sec^{-1} x = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}\csc^{-1} x = \frac{-1}{x\sqrt{x^2-1}}$$

Example 8

Differentiate $y = \dfrac{1}{\sin^{-1} x}$

Example 9

Differentiate and simplify $y = x \arctan \sqrt{x}$

Example 10

For $y^2 = xy + 3$

1. Find y'
2. Find the equations of all tangent lines going through points with $x = 2$

Example 11

Find y' for $\sin(xy + 1) = y^3 + x^2$

Example 12

Differentiate and simplify $y = x \sec^{-1}(3\sqrt{x})$

Example 13

Derive the formula for $\frac{d}{dx} \sec^{-1} x$
Hint: after differentiating, use trig identities to turn everything back into secants.