2.3 Calculating Limits Using Limit Laws

Graphs and tables of values can be used to estimate the limit of a function, but these techniques have serious limitations. They are computationally expensive and can lead to unreliable results.

A far more superior method for evaluating the limit of a function consists of using limit laws.

Result 1: Limit Laws

Suppose that c is a constant, that $n \in \mathbb{Z}^{>0}$, and that both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

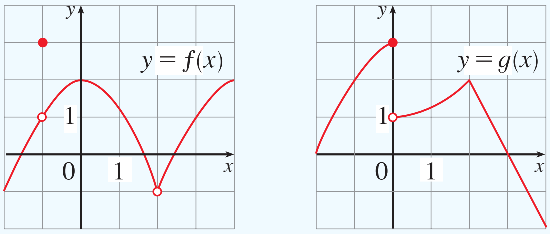
1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$ **Sum Law**
2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$ **Difference Law**
3. $\lim_{x\to a}[c\cdot f(x)] = c\cdot \lim_{x\to a} f(x)$ **Constant Multiple Law**
4. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$ **Product Law**
5. $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$ **Quotient Law**
6. $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$ **Power Law**
7. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ **Root Law**

   If n is even, then we assume that $\lim_{x\to a} f(x) > 0$
8. $\lim_{x\to a} c = c$
9. $\lim_{x\to a} x = a$
10. $\lim_{x\to a} x^n = a^n$
11. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$

    If n is even, then we assume that $\lim_{x\to a} f(x) > 0$

Example 1: 2.2.3

The graphs of f and g are given. Use them to evaluate each limit if it exists. If the limit does not exist, explain why.



a. $\lim_{x \to 2}[f(x) + g(x)]$

b. $\lim_{x \to 0}[f(x) - g(x)]$

c. $\lim_{x \to -1}[f(x)g(x)]$

d. $\lim_{x \to 3} \frac{f(x)}{g(x)}$

e. $\lim_{x \to 2}[x^2 f(x)]$

f. $f(-1) + \lim_{x \to -1} g(x)$

Solution

Example 2: 2.3.4

Evaluate the limit and state the laws that are being applied.

$$\lim_{x \to -3} (2x^3 + 6x^2 - 9)$$

Solution

Example 3: 2.3.6

Evaluate the limit and state the laws that are being applied.

$$\lim_{t \to 7} \frac{3t^2 + 1}{t^2 - 5t + 2}$$

Solution

Example 4: 2.3.8

Evaluate the limit and state the laws that are being applied.

$$\lim_{x \to 3} \sqrt[3]{x+5} \cdot (2x^3 - 3x)$$

Solution

The Direct Substitution Property

If $f(x)$ is either a polynomial or a rational function, then the limit as x approaches a is the same as if we were to directly evaluate a into the function (i.e. $\lim_{x \to a} f(x) = f(a)$).

Result 2: Direct Substitution

If f is a polynomial or rational function, and $a \in D_f$ then

$$\lim_{x \to a} f(x) = f(a)$$

Example 5

Let $f(x) = 5x^3 - 3x^2 + x - 6$. Determine $\lim_{x \to 3} f(x)$

Solution

Example 6

Let $f(x) = \frac{x+3}{x+6}$. Determine $\lim_{x \to 5} f(x)$

Solution

Example 7: 2.3.6

Evaluate the limit if it exists.

$$\lim_{x \to 6} \left(8 - \frac{1}{2}x \right)$$

Solution

Example 8: 2.3.10

Evaluate the limit if it exists.

$$\lim_{x \to -4} \frac{x^2 + 3x}{x^2 - x - 12}$$

Solution

Indeterminate Forms

Sometimes, direct substitution results in

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad -\frac{\infty}{\infty}, \quad \text{or} \quad \infty - \infty$$

These are known as **indeterminate forms**.

Indeterminate forms can be resolved by: factoring the expression; multiplying the expression by its conjugate, creating a common denominator; and/or using a combination of these techniques.

Example 9: 2.3.18

Evaluate the limit if it exists.

$$\lim_{x \to -5} \frac{2x^2 + 9x - 5}{x^2 - 25}$$

Solution

Example 10: 2.3.20

Evaluate the limit if it exists.

$$\lim_{u \to -1} \frac{u + 1}{u^3 + 1}$$

Solution

Example 11: 2.3.22

Evaluate the limit if it exists.

$$\lim_{x\to 9} \frac{9-x}{3-\sqrt{x}}$$

Solution

Example 12: 2.3.24

Evaluate the limit if it exists.

$$\lim_{x\to} \frac{2-x}{\sqrt{x+2}-2}$$

Solution

Example 13: 2.3.24

Evaluate the limit if it exists.

$$\lim_{h \to 0} \frac{(-2+h)^{-1} + 2^{-1}}{h}$$

Solution

Example 14: 2.3.26

Evaluate the limit if it exists.

$$\lim_{t \to 0} \left(\frac{1}{t} - \frac{1}{t^2 + t} \right)$$

Solution

Example 15

Show that $\lim_{x \to 0} |x| = dne$

Solution

Example 16: 2.3.44

Find the limit, if it exists. If the limit does not exist, explain why.

$$\lim_{x \to -4} \frac{|x+4|}{2x+8}$$

Solution

Example 17: 2.3.46

Find the limit, if it exists. If the limit does not exist, explain why.

$$\lim_{x \to -2} \frac{2 - |x|}{2 + x}$$

Solution

Example 18: 2.3.48

Find the limit, if it exists. If the limit does not exist, explain why.

$$\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{|x|} \right)$$

Solution

Additional Properties of Limits

The following two Theorems are helpful for evaluating the limit of a function.

Theorem 1

Let $f(x)$ and $g(x)$ be two functions, such that

$$f(x) \leq g(x)$$

when x is near a (except possibly at a). If both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$$

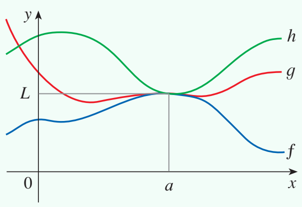
Theorem 2: Squeeze Theorem

Let $f(x)$, $g(x)$, and $h(x)$ be three functions such that

$$f(x) \leq g(x) \leq h(x)$$

when x is near a (except possibly at a) and if

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$



then

$$\lim_{x\to a} g(x) = L$$

Remark
The Squeeze Theorem says that if $g(x)$ is squeezed between $f(x)$ and $h(x)$, and both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} h(x)$ have the same limit as x approaches a, then $g(x)$ is forced to have the same limit as the two other functions that it is pinched between.

Example 19: 2.3.40

If $2x \leq g(x) \leq x^4 - x^2 + 2$ for all x, evaluate $\lim_{x \to 1} g(x)$

Solution

Example 20

Show that $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$

Solution

Example 21: 2.3.42

Prove that $\lim_{x \to 0^+} \sqrt{x} e^{\sin(\pi/x)} = 0$

Solution