2.5 Continuity

The word *continuous* in common parlance and refers to anything that has no breaks or gaps in it.

Similarly, in mathematics, a **continuous function** is a real valued function that has no holes, no gaps, no breaks, and no jumps in it.

Drawing the graph of a function is one way to determine if a function is continuous or not on an interval; but this is not always feasible. To overcome these issues, a technical definition is needed.

There are three types continuity: continuity at a single point, continuity from one-side, and continuity on an interval.

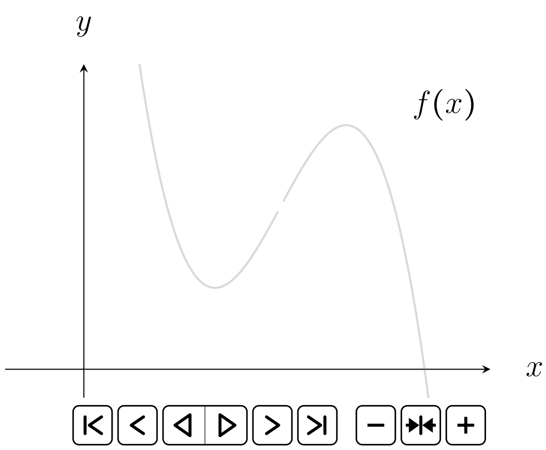
Continuity at a Single Point

Definition 1: Continuity at a Point

A function f is **continuous at the point** $x = a$ iff

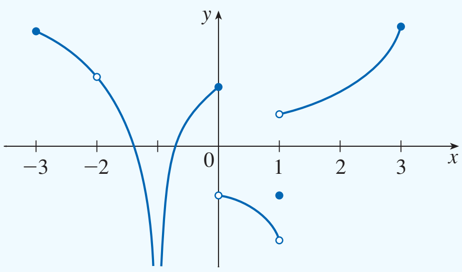
$$\lim_{x \to a} f(x) = f(a)$$

In order for a function to be continuous at the point, $x = a$, the general limit must exist (implying that $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$), and that number must also be equal to $f(a)$.



Example 1: 2.5.4

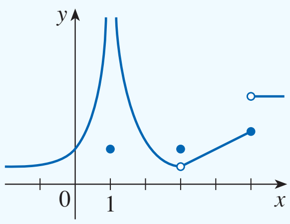
From the given graph of g, state the numbers at which g is discontinuous and explain why.



Solution

Example 2: 2.5.6

The graph of a function f is given.



a. At what numbers a does $\lim_{x \to a} f(x)$ not exist?

b. At what numbers a is f not continuous?

c. At what numbers a does $\lim_{x \to a} f(x)$ exist but f is not continuous at a?

Solution

Example 3: 2.5.14

Use the definition of continuity and the properties of limits to show that the function is continuous at the given number a.

$$g(t) = \frac{t^2 + 5t}{2t + 1} \quad , \quad a = 2$$

Solution

Example 4: 2.5.16

Use the definition of continuity and the properties of limits to show that the function is continuous at the given number a.

$$f(r) = \sqrt[3]{4r^2 - 2r + 7} \quad , \quad a = -2$$

Solution

Example 5: 2.5.20

Explain why the function is discontinuous at the given number a. Sketch the graph of the function.

$$f(x) = \begin{cases} \frac{1}{x+2} & ; \quad x \neq -2 \\ 1 & ; \quad x = -2 \end{cases} \qquad a = -2$$

Solution

Example 6: 2.5.22

Explain why the function is discontinuous at the given number a. Sketch the graph of the function.

$$f(x) = \begin{cases} \frac{x^2-x}{x^2-1} & ; \quad x \neq 1 \\ 1 & ; \quad x = 1 \end{cases} \qquad a = 1$$

Solution

Example 7: 2.5.24

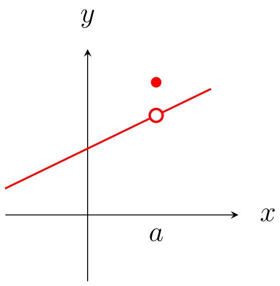
Explain why the function is discontinuous at the given number a. Sketch the graph of the function.

$$f(x) = \begin{cases} \frac{2x^2-5x-3}{x-3} & ; \quad x \neq 3 \\ 6 & ; \quad x = 3 \end{cases} \qquad a = 3$$

Solution

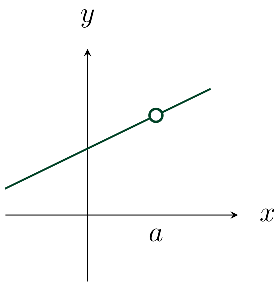
Types of Discontinuity

There are three types of discontinuities: removable, infinite, and jump discontinuities.



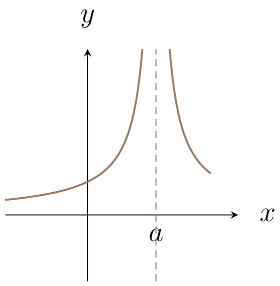
Removable

$\lim_{x\to a} f(x) \neq f(a)$



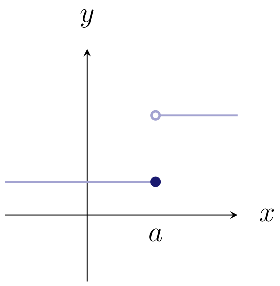
Removable

$\lim_{x\to a} f(x) = L$
$f(a) = und$



Infinite

$\lim_{x\to a} f(x) = \infty$



Jump

$\lim_{x\to a-} f(x) \neq \lim_{x\to a+} f(x)$

Definition 2: One Sided Continuity

A function f is said to be

- **continuous from the right** of a if $\lim_{x\to a+} f(x) = f(a)$, and
- **continuous from the left** of a if $\lim_{x\to a-} f(x) = f(a)$

Example 8: 2.5.44

Find the numbers at which f is discontinuous. At which of these numbers is f continuous from the right, from the left, or neither? Sketch the graph of f.

$$f(x) = \begin{cases} 2^x & ; \quad x \leqslant 1 \\ 3 - x & ; \quad 1 < x \leqslant 4 \\ \sqrt{x} & ; \quad x > 4 \end{cases}$$

Solution

Example 9: 2.5.45

Find the numbers at which f is discontinuous. At which of these numbers is f continuous from the right, from the left, or neither? Sketch the graph of f.

$$f(x) = \begin{cases} x+2 & ; \quad x < 0 \\ e^x & ; \quad 0 \leqslant x \leqslant 1 \\ 2-x & ; \quad x > 1 \end{cases}$$

Solution

Continuity on an Interval

Definition 3: Continuity on an Interval

A function f is **continuous on an interval** if it is continuous at every number in the interval.

If f is defined only on one side of an endpoint of the interval then *continuous* at the endpoint is understood to be a one-sided continuity.

Example 10: 2.5.17

Use the definition of continuity and the properties of limits to show that the function is continuous on the given interval.

$$f(x) = x + \sqrt{x-4}, \quad [4, \infty)$$

Solution

Example 11: 2.5.18

Use the definition of continuity and the properties of limits to show that the function is continuous on the given interval.

$$g(x) = \frac{x-1}{3x+6}, \quad (-\infty, -2)$$

Solution

Continuity, is a fundamental property that allows for the application of key mathematical theorems (e.g. differentiability, assertion of intermediate values, etc).

The following theorems illustrate how properties of continuity follows from those of limits.

Theorem 1

If f and g are two continuous functions at a and $c \in \mathbb{R}$ is a constant, then the following functions are also continuous at a

$$f+g \qquad f-g \qquad cf \qquad f\times g \qquad \frac{f}{g} \quad ; \quad g(a)\neq 0$$

Theorem 2

The following types of functions are continuous at every number in their domains

- Polynomials
- Rational functions
- Root functions
- Trigonometric functions
- Inverse trigonometric functions
- Exponential functions
- Logarithmic functions

Theorem 3

If f is continuous at b and $\lim_{x\to a} g(x) = b \quad \Rightarrow \quad \lim_{x\to a} f(g(x)) = f(b)$

That is

$$\lim_{x\to a} f(g(x)) = f\left[\lim_{x\to a} g(x)\right] = f(b)$$

Theorem 4

If g is continuous at a and f is continuous at $g(a)$, then the composite function

$$f\circ g = (f\circ g)(x) = f(g(x))$$

is also continuous at a.

Example 12: 2.5.28

Explain, using Theorems 1, 2, 3, and 4 , why the function is continuous at every number in its domain. State the domain.

$$g(v) = \frac{3v - 1}{v^2 + 2v - 15}$$

Solution

Example 13: 2.5.30

Explain, using Theorems 1, 2, 3, and 4 , why the function is continuous at every number in its domain. State the domain.

$$B(u) = \sqrt{3u - 2} + \sqrt[3]{2u - 3}$$

Solution

Example 14: 2.5.32

Explain, using Theorems $1, 2, 3$, and 4 , why the function is continuous at every number in its domain. State the domain.

$$f(t) = e^{-t^2} \ln\left(1 + t^2\right)$$

Solution

Example 15: 2.5.34

Explain, using Theorems $1, 2, 3$, and 4 , why the function is continuous at every number in its domain. State the domain.

$$g(t) = \cos^{-1}\left(e^t - 1\right)$$

Solution

Example 16: 2.5.36

Use continuity to evaluate the limit

$$\lim_{\theta \to \pi/2} \sin(\tan(\cos\theta))$$

Solution

Example 17: 2.5.38

Use continuity to evaluate the limit

$$\lim_{x \to 4} 3^{\sqrt{x^2-2x-4}}$$

Solution

Example 18: 2.5.48

Find the values of a and b that make f continuous everywhere.

$$f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \leqslant x < 3 \\ 2x - a + b & \text{if } x \geqslant 3 \end{cases}$$

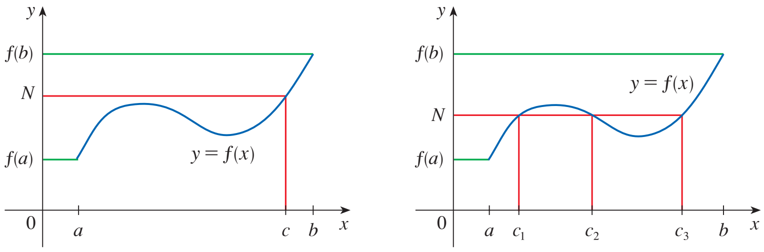
Solution

The Intermediate Value Theorem

The Intermediate Value Theorem is a mathematical statement concerning the property of a continuous function.

Theorem 5: The Intermediate Value Theorem

Suppose that f is a continuous function on the closed interval $[a, b]$, and let N be any number between $f(a)$ and $f(b)$ such that $f(a) \neq f(b)$. Then, $\exists c \in (a, b)$ such that $f(c) = N$.



In other words, given a smooth, unbroken curve (a continuous function) that starts at one point and ends at another, then the curve will pass through every value in between those two points at least once.

Example 19: 2.5.53

f $f(x) = x^2 + 10 \sin x$, show that there is a number c such that $f(c) = 1000$.

Solution

Example 20: 2.5.56

Use the Intermediate Value Theorem to show that there is a solution to the given equation in the specified interval.

$$\ln x = x - \sqrt{x} \quad ; \quad (2, 3)$$

Solution

Example 21: 2.5.56

Use the Intermediate Value Theorem to show that there is a solution to the given equation in the specified interval.

$$\sin x = x^2 - x \quad ; \quad (1, 2)$$

Solution