

2.6: Limits at Infinity; Horizontal Asymptotes

Limits at infinity describe the behaviour of a function as the input variable approaches positive or negative infinity. Essentially, we want to investigate how the function behaves as x becomes arbitrarily large (i.e. $x \rightarrow \infty$) or x becomes arbitrarily small (i.e. $x \rightarrow -\infty$)

For clarity:

- *Infinite limits* refer to $\lim_{x \rightarrow a} f(x) = -\infty$ and $\lim_{x \rightarrow a} f(x) = \infty$
- *Limits at infinity* refer to $\lim_{x \rightarrow -\infty} f(x)$ and $\lim_{x \rightarrow \infty} f(x)$

Limits At Infinity

Definition 1: Limits at Positive Infinity

Let f be a function defined on (a, ∞) . Then

$$\lim_{x \rightarrow \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large and positive.

Definition 2: Limits at Negative Infinity

Let f be a function defined on $(-\infty, a)$. Then

$$\lim_{x \rightarrow -\infty} f(x) = L$$

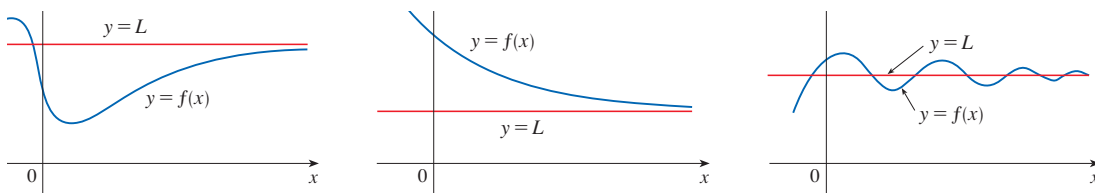
means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large and negative.

Definition 3: Horizontal Asymptotes

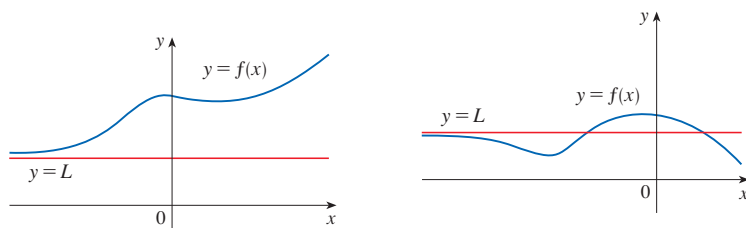
The line $y = L$ is called the **horizontal asymptote** of $y = f(x)$ if

$$\lim_{x \rightarrow -\infty} f(x) = L \quad \text{or} \quad \lim_{x \rightarrow +\infty} f(x) = L$$

The graphs below illustrate some examples of $\lim_{x \rightarrow \infty} f(x) = L$ along with their horizontal asymptotes.

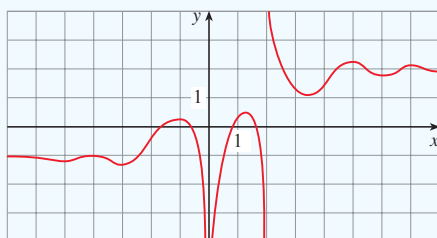


The graphs below illustrate some examples of $\lim_{x \rightarrow -\infty} f(x) = L$ along with their horizontal asymptotes.



Example 1: 2.6.4

For the function g whose graph is given below.



State the following:

- $\lim_{x \rightarrow \infty} g(x)$
- $\lim_{x \rightarrow -\infty} g(x)$
- $\lim_{x \rightarrow 0} g(x)$
- $\lim_{x \rightarrow 2^-} g(x)$
- $\lim_{x \rightarrow 2^+} g(x)$
- The equations of the asymptotes

Solution

Evaluating Limits At Infinity

In general, Limit Laws cannot be applied to limits at infinity because ∞ is not a number. However, the following result is extremely useful when evaluating limits at infinity.

Theorem 1

If $r \in \mathbb{Q}^{>0}$ then $\lim_{x \rightarrow \infty} \frac{1}{x^r} = 0$

If $r \in \mathbb{Q}^{>0}$ such that x^r is defined $\forall x$ then $\lim_{x \rightarrow -\infty} \frac{1}{x^r} = 0$

The first part of the theorem says, is that if the denominator grows without bound, but the numerator remains fixed, then the limit of $\frac{1}{x^r} \rightarrow 0$ as $x \rightarrow +\infty$. The second part just adds a provision for x^r to make sense when $x \rightarrow -\infty$.

Example 2

Evaluate the limit or show that it does not exist

$$\lim_{x \rightarrow \infty} (x^5 - x^4)$$

Solution

Example 3

Evaluate the limit or show that it does not exist

$$\lim_{x \rightarrow -\infty} (2x^3 + 2x^2 + x - 1)$$

Solution

Remark

Theorem 1 by itself, may not be sufficient to evaluate the limit at hand. If an indeterminate form such as $\infty - \infty$ or $\frac{\infty}{\infty}$ arises, we can resolve these by factoring, multiplying by conjugates, making common denominators, and or applying the Squeeze Theorem.

Example 4: 2.6.16

Evaluate the limit or show that it does not exist.

$$\lim_{x \rightarrow \infty} -\frac{2}{3x + 7}$$

Solution**Example 5: 2.6.18**

Evaluate the limit or show that it does not exist

$$\lim_{t \rightarrow -\infty} \frac{6t^2 + t - 5}{9 - 2t}$$

Solution

Example 6: 2.6.18

Evaluate the limit or show that it does not exist

$$\lim_{u \rightarrow \infty} \frac{(u^2 + 1)(2u^2 - 1)}{(u^2 + 2)^2}$$

Solution**Example 7: 2.6.28**

Evaluate the limit or show that it does not exist $\lim_{q \rightarrow \infty} \frac{q^3 + 6q - 4}{4q^2 - 3q + 3}$

Solution

The following few examples rely on the following result for $\sqrt{x^2}$ and the formal definition of $|x|$.

Result 1

Let $x \in \mathbb{R}$. Then,

$$\sqrt{x^2} = |x|$$

Definition 4: Absolute Value

Let $x \in \mathbb{R}$. Then,

$$|x| = \begin{cases} -x & ; & x < 0 \\ 0 & ; & x = 0 \\ +x & ; & x > 0 \end{cases}$$

Example 8

Evaluate the limit or show that it does not exist

$$\lim_{x \rightarrow \infty} \frac{\sqrt{9x^2 - 6}}{x + 1}$$

Solution

Example 9

Evaluate the limit or show that it does not exist

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{9x^2 - 6}}{x + 1}$$

Solution

Example 10: 2.6.24

Evaluate the limit or show that it does not exist

$$\lim_{t \rightarrow \infty} \frac{t + 3}{\sqrt{2t^2 - 1}}$$

Solution

Example 11: 2.6.26

Evaluate the limit or show that it does not exist

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{1 + 4x^6}}{2 - x^3}$$

Solution

Example 12: 2.6.32

Evaluate the limit or show that it does not exist

$$\lim_{x \rightarrow \infty} (x - \sqrt{x})$$

Solution

Example 13: 2.6.30

Evaluate the limit or show that it does not exist

$$\lim_{x \rightarrow \infty} (\sqrt{4x^2 + 3x} + 2x)$$

Solution

Example 14: 2.6.34

Evaluate the limit or show that it does not exist

$$\lim_{x \rightarrow \infty} (e^{-x} + 2 \cos(3x))$$

Solution**Example 15: 2.6.40**

Evaluate the limit or show that it does not exist

$$\lim_{x \rightarrow 0^+} \arctan(\ln x)$$

Solution

Example 16: 2.6.36

Evaluate the limit or show that it does not exist

$$\lim_{x \rightarrow \infty} \frac{\sin^2 x}{x^2 + 1}$$

Solution**Example 17: 2.6.41**

Evaluate the limit or show that it does not exist

$$\lim_{x \rightarrow \infty} [\ln(x^2 + 1) - \ln(x + 1)]$$

Solution

Example 18: 2.6.48

Find the vertical and horizontal asymptotes for the function

$$y = \frac{2x^2 + 1}{3x^2 + 2x - 1}$$

Solution

Example 19: 2.6.50

Find the vertical and horizontal asymptotes for the function

$$y = \frac{1 + x^4}{x^2 - x^4}$$

Solution**Example 20: 2.6.52**

Find the vertical and horizontal asymptotes for the function

$$y = \frac{2e^x}{e^x - 5}$$

Solution

Example 21: 2.6.44

a. For $f(x) = \frac{2}{x} - \frac{1}{\ln x}$ find each of the following limits.

i. $\lim_{x \rightarrow \infty} f(x)$

ii. $\lim_{x \rightarrow 0^+} f(x)$

iii. $\lim_{x \rightarrow 1^-} f(x)$

iv. $\lim_{x \rightarrow 1^+} f(x)$

b. Use the information from (a) to make a rough sketch of $f(x)$.

Solution

Example 22: 2.6.60

Find the limits of the function as $x \rightarrow -\infty$ and as $x \rightarrow +\infty$. Use this information along with the intercepts of the function to make a rough sketch of the graph.

$$y = 2x^3 - x^4$$

Solution

Example 23: 2.6.62

Find the limits of the function as $x \rightarrow -\infty$ and as $x \rightarrow +\infty$. Use this information along with the intercepts of the function to make a rough sketch of the graph.

$$y = x^3(x + 2)^2(x - 1)$$

Solution