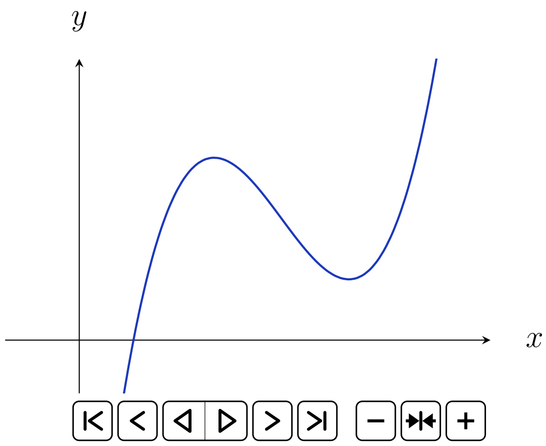
2.7: Derivatives and Rates of Change

Suppose that we have a function, $f(x)$. To find the slope of the tangent line to the curve at some fixed point, $P = (a, f(a))$, we would

1. Choose a point $Q = (x, f(x))$ on the curve.
2. Calculate the slope of the secant line that contains (joins) P and Q.
3. Move the point Q closer to P, draw another secant line, calculate its slope, and repeat.

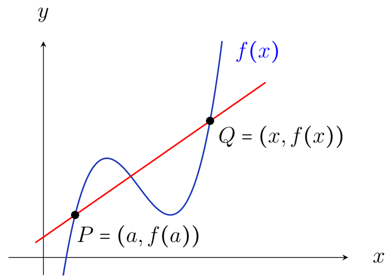


Observe that, as Q moves closer to P, the distance between P and Q shrinks, and the slopes of the secant lines approaches the slope of the tangent line in value.

Thus, if m_T is the slope of the tangent line, at the point $P = (a, f(a))$, then

$$\begin{aligned} m_T &= \lim_{Q \to P} m_{PQ} \\ &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \end{aligned}$$

Tangent Lines



The slope of the tangent line is the limit of the slopes of the secant lines.

> **Definition 1: Tangent Line**
>
> The **tangent line** to the curve $y = f(x)$ at the point $P = (a, f(a))$ is the line through P with slope
>
> $$m = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$
>
> provided that the limit exists.

The definition above provides us with a formula to calculate the slope of the tangent line at the point P. However, many situations require the equation of the tangent line as well. The equation of a line can be found using

- Slope-intercept form: $y = mx + b$
- Point-slope form: $y - y_1 = m \cdot (x - x_1)$

> **Definition 2: Derivative of a Function at a**
>
> The slope of the tangent line to the curve $f(x)$ at the point $x = a$ is called the **derivative of the function at** a. It is denoted as $f'(a)$ and read as *"f prime at a"*.

Since the slope of the tangent line and the derivative of the function are the same thing, the m in the formula in **Definition 1**, could be replaced with $f'(a)$

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

The same could be said about replacing the m in the point slope form of the line with $f'(a)$.

$$y - y_1 = m \cdot (x - x_1)$$
$$y - f(a) = f'(a) \cdot (x - a)$$

Example 1: 2.7.6

Find an equation of the tangent line to the curve at the given point.

$$y = x^2 - 2x^3 \qquad ; \qquad (1, -1)$$

Solution

Example 2: 2.7.8

Find an equation of the tangent line to the curve at the given point.

$$y = \sqrt{1 - 3x} \qquad ; \qquad (-1, 2)$$

Solution

Example 3: 2.7.10

a. Find the slope of the tangent line to the curve $y = 2\sqrt{x}$ at the point $x = a$

b. Find the equation of the tangent lines at the points $(1, 2)$ and $(9, 6)$

c. Graph the curve and both tangent lines on the same axes.

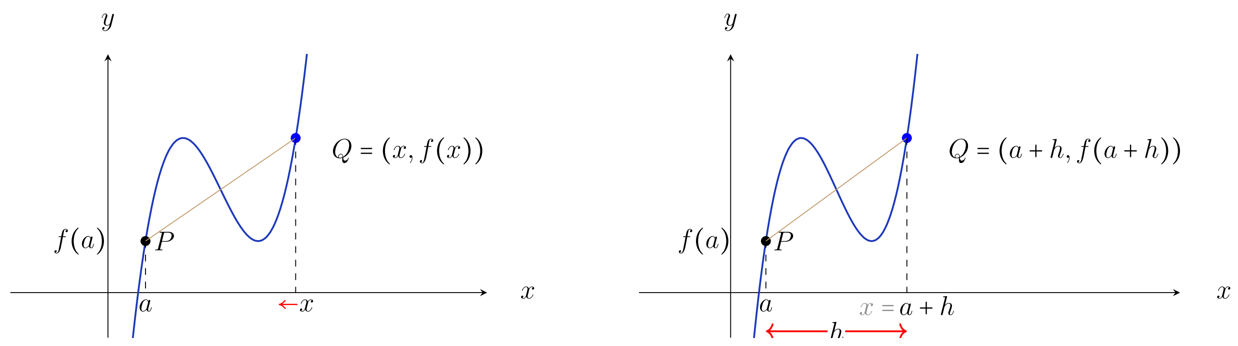
Solution

Definition of the Derivative

The slope of the tangent line at $x = a$ is given by $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.

However, this is not the most convenient way to compute the derivative the function.

The two graphs below shows that if $P = (a, f(a)$ and $Q = (x, f(x))$ are two points on the curve $y = f(x)$, and h is the distance between a and x, then x is really $x = a + h$, and $f(x)$ can be relabelled as $f(a+h)$. As $x \to a$, the distance between the two points, $h \to 0$.



By relabelling x to $a + h$, we can have the following definition of the derivative.

Definition 3: Derivative of a Function

The **derivative of a function at** a, denoted as $f'(a)$ and is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

The following calculations show that the formulas in Definition 1 and Definition 3 are equivalent:

$$\begin{aligned} m = f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\ &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{(a+h) - a} \\ &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \end{aligned}$$

And, a more elegant way to write the equation of the tangent line is as follows

$$\begin{aligned} y - y_1 &= m \cdot (x - x_1) \\ y - f(a) &= f'(a)(x - a) \end{aligned}$$

Example 4: 2.7.20

Using the definition of the derivative, find $f'(a)$ for

$$f(x) = 5x^4 \qquad \text{at} \qquad a = -1$$

Solution

Example 5: 2.7.22

Using the definition of the derivative, find the $f'(a)$ for

$$f(x) = \frac{1}{\sqrt{2x+2}} \qquad \text{at} \quad a = 1$$

Solution

Example 6: 2.7.24

Find $f'(a)$ using the definition of the derivative if $f(t) = t^3 - 3t$

Solution

Example 7: 2.7.26

Find $f'(a)$ using the definition of the derivative if $f(x) = \frac{x}{1-4x}$

Solution

Example 8: 2.7.28

Find the equation of the tangent line of the graph $y = g(x)$ at $x = 5$ if $g(5) = -3$ and $g'(5) = 4$

Solution

Example 9: 2.7.44

Each limit represents the derivative of some function f at some number a. State such an f and a in each case.

$$\lim_{h \to 0} \frac{e^{-2+h} - e^{-2}}{h}$$

Solution

Example 10: 2.7.44

Each limit represents the derivative of some function f at some number a. State such an f and a in each case.

$$\lim_{x \to 1/4} \frac{\frac{1}{x} - 4}{x - \frac{1}{4}}$$

Solution

Example 11: 2.7.44

Each limit represents the derivative of some function f at some number a. State such an f and a in each case.

$$\lim_{\theta \to \pi/6} \frac{\sin \theta - \frac{1}{2}}{\theta - \frac{\pi}{6}}$$

Solution

Example 12

Determine if $f'(0)$ exists if

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & ; \quad x \neq 0 \\ 0 & ; \quad x = 0 \end{cases}$$

Solution

Average vs. Instantaneous Rate of Change

The **slope of the secant line** represents the **average rate** of change and the **slope of the tangent line** corresponds to the **instantaneous rate** of change.

Remark
The slope of the tangent line, the derivative of the function, and the instantaneous rate of change, mean the same thing.

The following examples illustrate how these concepts are applied in different applications.

Example 13

The displacement (in feet) of a particle moving in a straight line is given by $s(t) = t^2 - 8t + 18$, where t is measured in seconds.

a. Find the average velocity over each time interval:

   i. $[3, 4]$
   ii. $[3.5, 4]$
   iii. $[4, 5]$
   iv. $[4, 4.5]$

b. Find a formula for the instantaneous velocity, $v(t)$.

c. Find the instantaneous velocity when $t = 4$.

Solution

Example 14

If a rock is thrown upward on the planet Mars with a velocity of $10\,m/s$ its height (in meters) after t seconds is given by

$$H(t) = 10t - 1.86t^2$$

a. Find the velocity of the rock after one second.

b. Find the velocity of the rock when $t = a$.

c. When will the rock hit the surface?

d. With what velocity will the rock hit the surface?

Solution

Example 15: 2.7.49

The cost (in dollars) of producing x units of a certain commodity is

$$C(x) = 5000 + 10x + 0.05x^2$$

a. Find the average rate of change of C with respect to x when the production level is changes

   i. from $x = 100$ to $x = 105$

   ii. from $x = 100$ to $x = 101$

b. Find the instantaneous rate of change of C with respect to x when $x = 100$, this is called the **marginal cost**.

Solution