2.8: The Derivative as a Function

The **derivative of a function at** $x = a$ can be computed by

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \qquad \text{or} \qquad f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Geometrically, $f'(a)$ is the slope of the tangent line to the curve of $y = f(x)$ at $(a, f(a))$. The derivative represents the instantaneous rate of change of y with respect to x at the point, $x = a$.

In both of these formulations the value of a is fixed. In this section, we examine what happens when the value of a is allowed to change.

Derivative of as a Function

Suppose that $f(x)$ is a function, and we want to calculate the derivative of f at x. To do so, simply replace a with x in the definition of the derivative to generate a new formulation.

Definition 1: The Derivative with Respect to x

The **derivative of $f(x)$ with respect to x** is the function $f'(x)$ and is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We say that f itself is differentiable at x, and that f has a derivative whenever the limit exists.

Remark
Observe that $f'(x)$ is a function. Therefore, it has a domain and a range.

The expression above provides a convenient means for calculating the derivative of the function, at any point in its domain (provided that the limit exists).

Geometric Interpretation of the First Derivative of a Function

The derivative of a function is the instantaneous rate of change of y with respect to x.

Remark The statement "with respect to" is often abbreviated to w.r.t

Result 1: Intervals of Increasing/Decreasing

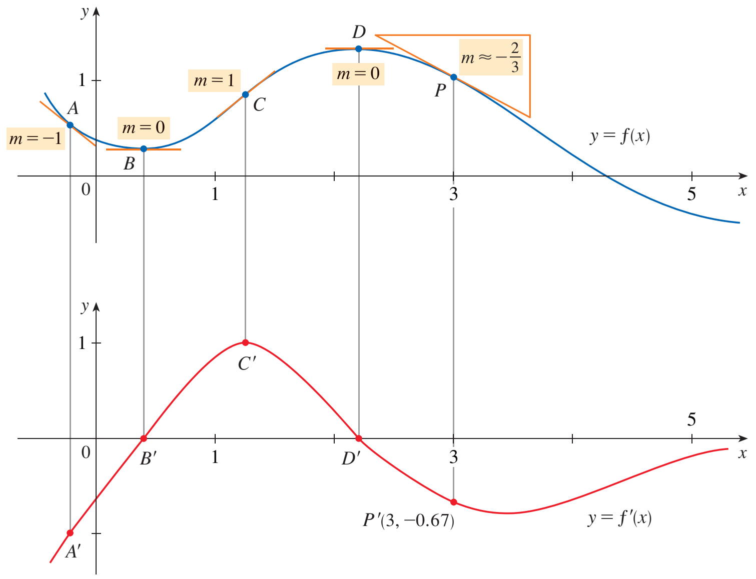
Geometrically, the value of $f'(x)$ tells us whether the function is increasing, decreasing, or flat at a specific value of x, or on an a given interval.

$f'(x) > 0 \quad \Rightarrow \quad f$ is increasing.

$f'(x) < 0 \quad \Rightarrow \quad f$ is decreasing.

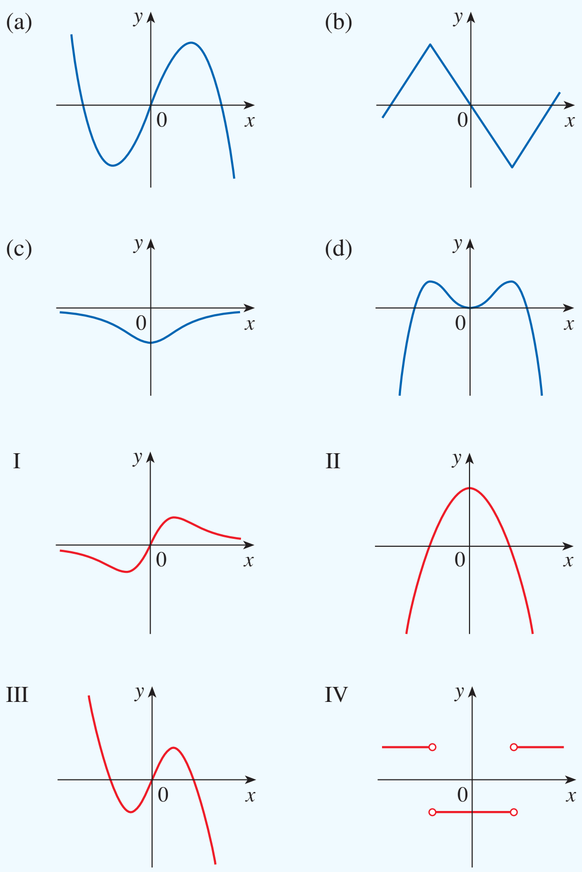
$f'(x) = 0 \quad \Rightarrow \quad f$ is neither increasing nor decreasing; so it is flat.

The graphs below show how the derivative of a function, is the slope of the tangent line to $f(x)$ at a. Observe that whenever the $f(x)$ is increasing, $f'(x)$ is above the $x-$axis. This is because whenever $f(x)$ is increasing, the slopes of the tangent lines will be positive; and hence, $f'(x)$ will be positive. Similarly, whenever $f(x)$ is decreasing; $f'(x)$ is below the $x-$axis.



Example 1: 2.8.3

Match the graph of each function in (a)-(d) with the graph of its derivative in I-IV. Give reasons for your choices.



Solution

Example 2: 2.8.22

Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

$$f(x) = mx + b$$

Solution

Example 3: 2.8.24

Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

$$f(x) = 4 + 8x - 5x^2$$

Solution

Example 4: 2.8.28

Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

$$f(v) = \frac{v}{v+2}$$

Solution

Example 5: 2.8.32

Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

$$g(x) = \frac{1}{1 + \sqrt{x}}$$

Solution

Example 6: 2.8.33

a. Sketch the graph of $f(x) = 1 + \sqrt{x+3}$ by starting with the graph of $y = \sqrt{x}$ and using transformations.

b. Use the graph from part (a) to sketch the graph of $f'(x)$.

c. Use the definition of a derivative to find $f'(x)$.

d. What are the domains of $f(x)$ and $f'(x)$?

Solution

Higher Derivatives

Let f be a differentiable function. Then its derivative, f', is also a function. This implies that f' may also have a derivative of its own, (provided that certain conditions are met).

The derivative of f' is denoted as f'' and is called the **second derivative of** f. The same could be said about f'' having its own derivative, f''', and so on.

Notation

The two most frequently used notational systems in calculus are: Lagrange (prime) notation, and Leibniz notation.

Prime notation

$$y' = f'(x) \qquad y'' = f''(x) \qquad y''' = f'''(x) \qquad y^{(4)} = f^{(4)}(x) \qquad \cdots$$

Leibniz notation

$$y' = \frac{dy}{dx} \qquad y'' = \frac{d^2y}{dx^2} \qquad y''' = \frac{d^3y}{dx^3} \qquad y^{(4)} = \frac{d^4y}{dx^4} \qquad \cdots$$

Geometric Interpretation of the Second Derivative

The **first derivative** of a function tells us on what interval(s) the function is **increasing or decreasing**.

The **second derivative** tells us on what interval(s) the function is **concave (opening) up or concave (opening) down**.

In physics, the higher order derivatives are used to model the movement of objects. Let $s(t)$, model the displacement of an object at time, t. Then the

- velocity is the instantaneous rate of change of the displacement $w.r.t$ time:
$$v(t) = s'(t) = \frac{ds}{dt}$$
- acceleration is the instantaneous rate of change of the velocity $w.r.t$ time::
$$a(t) = v'(t) = s''(t) = \frac{d^2s}{dt^2}$$
- Jerk is the instantaneous rate of change of the acceleration $w.r.t$ time:
$$j(t) = a'(t) = v''(t)s'''(t) = \frac{d^3s}{dt^3}$$

Example 7

Let $f(x) = x^3 - 3x$. Use the definition of a derivative to find $f'(x), f''(x)$, and $f''(x)$. Then graph on the same axes to see if your answers are reasonable.

Solution

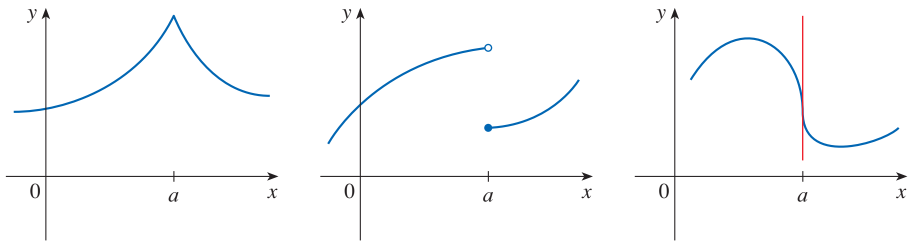
When is a Function Not Differentiable?

Not all functions are differentiable. A function is differentiable at $x = a$ iff the limit of

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists.

A function can fail to be differentiable for several reasons: they can have a cusp/corner at $x = a$, be discontinuous at $x = a$, or have a vertical tangent line at $x = a$



(a) A corner (b) A discontinuity (c) A vertical tangent

Definition 2: Differentiability at $x = a$

A function f is **differentiable at** a if $f'(a)$ exists.

It is differentiable on an open interval (a, b) [or (a, ∞), or $(-\infty, b)$, or$(-\infty, \infty)$] if it is differentiable at every number in the interval.

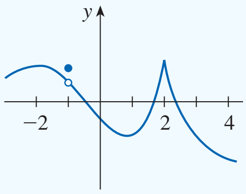
Theorem 1

If f is differentiable at a, then f is continuous at a.

Remark
The converse is not true. Continuity does not guarantee differentiability.

Example 8: 2.8.42

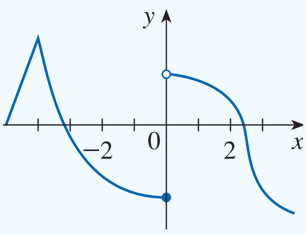
The graph of f is given. State, with reasons, the numbers at which f is not differentiable.



Solution

Example 9: 2.8.41

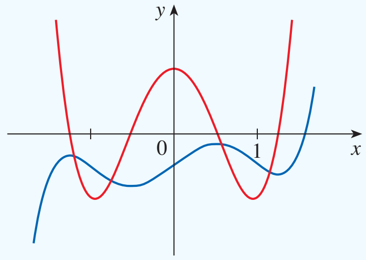
The graph of f is given. State, with reasons, the numbers at which f is not differentiable.



Solution

Example 10: 2.8.47

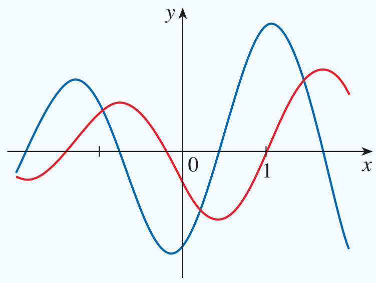
The graphs of a function f and its derivative $f'(x)$ are shown. Which is bigger, $f(1)$ or $f'(1)$?



Solution

Example 11: 2.8.48

The graphs of a function f and its derivative $f'(x)$ are shown. Which is bigger, $f(1)$ or $f'(1)$?



Solution

Example 12: 2.8.58

a. If $g(x) = x^{2/3}$ show that $g'(0)$ does not exist.

b. If $a \neq 0$ find $g'(a)$

c. Show that $g(x) = x^{2/3}$ has a vertical tangent line at $(0, 0)$

d. Illustrate part(c) by sketching a graph of $g(x) = x^{2/3}$

Solution

Example 13

Determine if $f(x) = |x|$ is differentiable at $x = 0$. Sketch both $f(x)$ and $f'(x)$

Solution

Example 14: 2.8.62

a. Sketch the graph of the function $f(x) = x + |x|$.

b. For what values of x is $f(x)$ differentiable?

c. Find a formula for f'.

Solution