1.1 Four Ways to Represent a Function

Functions are widely used in science and mathematics. Whenever we want to describe the relationship between several quantities or model the behaviours observed in a system, we use functions to achieve that.

Functions and Calculus

Calculus is the study of how things change. Before calculus was invented, all math was static. This meant that it could only be used to calculate objects which were perfectly still.

But the world is constantly moving - from the stars and planets in the solar system, to the blood pulsing through your veins - static mathematics was simply inadequate to model systems which undergo constant change. To address this issue, the theory of calculus was developed. The most basic objects that we deal with in calculus are functions.

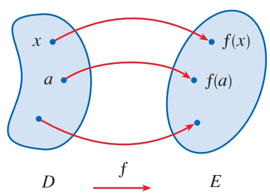
Functions are idealizations of how one quantity changes in relation to another quantity. For example, to describe

- the position of a planet as a function time
- the resale value of car in relation to the number of kilometres it has been driven
- the performance of the stock market as a measure of investors' confidence

Intuitively, a function is a process that associates an element from one set to an element in another set. Here is a more formal definition

> **Definition 1: Function**
>
> A **function**, f, is a rule that assigns to each element x in a set D, exactly one element, called $f(x)$, in set E.

Typically, these mappings are illustrated with arrow diagrams. In the picture below, the element x from set D is associated with the element $f(x)$, in E; the element a is associated with $f(a)$, and so on.



Definition 2: Domain

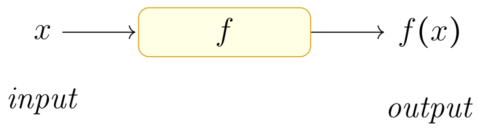
Set D is called the **domain** of the function. It is the set of all values for which the function is defined.

> **Remark**
> In this course, it is extremely important that you be able to state or work out the domain of a function simply by looking at its algebraic form. Being able to do this for the range is *not so* important; but for a small class of functions you should be able to do so; especially when presented with a graph.

Definition 3: Range

Set E is called the **range** of the function. This is the set of all values that the function can take on or assumes based on its domain.

It is helpful to think of functions as machines; one that accepts inputs labelled as, x, and produces outputs labelled as, $f(x)$, according to some rule, f.



So in this analogy, the domain of the function, is the set of all values that the machine can accept, and the range is the set of all values that the machine can produce.

Lastly, in the language of mathematics, the input and output values of a function are formally defined as follows.

Definition 4: Independent and Dependent Variables

The letter x is called the **independent variable** or **argument** of the function, and $f(x)$ is called the **dependent** variable or **value** of the function.

> **Remark**
> Although it is customary to denote functions with the letters f, g, and h, other letters and symbols can also be used (e.g. $\Phi, \Gamma, \xi, \ldots$). The same can be said about the letters used to designate independent variables (i.e. a function's argument need not be called 'x', other labels maybe used in its place).

Four Ways To Represent a Function

A function is a way of matching members from one set to another. There are four ways to represent this matching process: verbally (with a written description), numerically (with a table of values),visually (with a graph), and algebraically (with an equation).

Being able to covert a function between each of these forms, is an important skill in algebra and calculus; because additional information about the function can be gained from its different representations.

- Verbal Representation

  - When modelling a process mathematically, we always begin with a verbal description of the of problem.In this course, you will encounter verbal descriptions of functions in the form of word problems.

- Numerical Representation
  - In this representation, information about the mappings are conveyed through sets of ordered pairs. The ordered pairs can either be enumerated in a list, or presented in a table; which in turn can be used to produce a graphical or visual representation of the function

- Visual Representation
  - This involves modelling a function in a dimensional overlay (ie. plotting the ordered pairs in a Cartesian plane). A visual or graphical representation of a function provides us with the ability to study and analyse a function's behaviour.

- Algebraic Representation
  - This is the most common, most concise, and most powerful way to represent functions. The algebraic form of a function, is the mapping or rule of assignment made explicit with a formula, and is the most efficient means of determining the function's image given some argument.

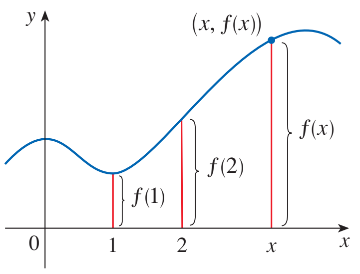
The Graph of a Function

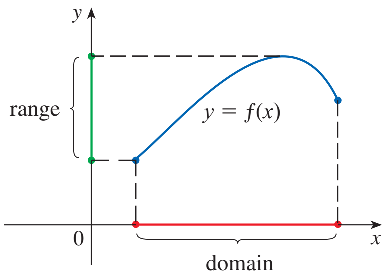
The most common way to visualize a function is with a graph. Below is the technical definition for the graph of a function.

> **Definition 5: Graph**
>
> If f is a function with domain D, then its **graph** is the set of all ordered pairs
>
> $$\{(x, f(x)) \,|\, x \in D\}$$

Simply put, when the domain and the range of a function are sets of real numbers, each pair $(x, f(x)) = (x, y)$ can be thought of as the Cartesian coordinates of a point in the plane. If all ordered pairs of the function are plotted in the plane, then resulting image is the graph of f.

The graph of a function, allows us to interpret $f(x)$ geometrically as the height of the function above the point at x. It also organizes the values of the domain along the x−axis and the values of the range along the y−axis.

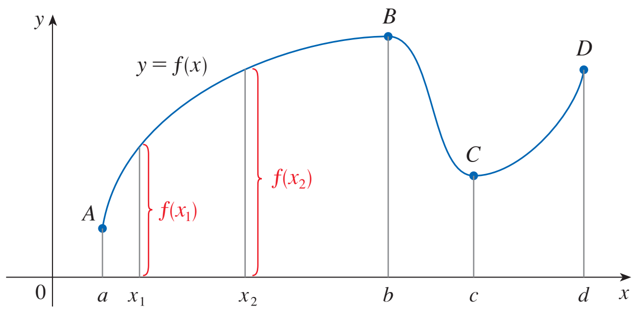




The graph also reveals the short term and long term behaviour of the function; i.e. intervals for which the function is increasing or decreasing, continuous or discontinuous, etc.

Definition 6: Intervals of Increasing and Decreasing

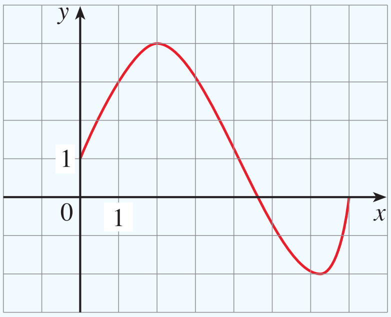
function f is called **increasing** on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I and said to be **decreasing** on I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in I.



So in the graph above, the function f is increasing on the intervals $[a, b] \cup [c, d]$ and decreasing on $[b, c]$

Example 1: 1.1.3

The graph of the function f is given below.



a. Find the values of $f(0), f(2), f(6)$ and $f(7)$

b. What is the domain and range of f?

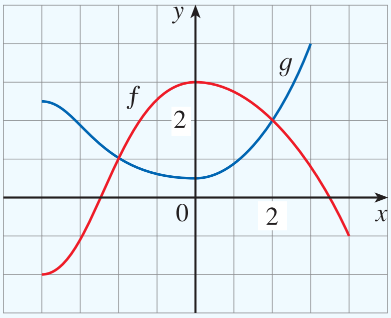
c. State the interval(s) on which f is decreasing.

d. State the interval(s) for which the function is increasing.

Solution

Example 2: 1.1.4

The graphs of f and g are given.



a. State the values of $f(-4)$ and $g(3)$.

b. Which is larger, $f(-3)$ or $g(-3)$?

c. For what values of x is $f(x) = g(x)$?

d. On what interval(s) is $f(x) \leq g(x)$?

e. State the solution of the equation $f(x) = -1$

f. On what interval(s) is g decreasing?

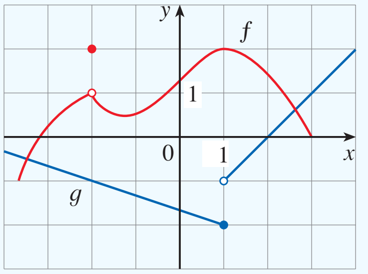
g. State the domain and range of f.

h. State the domain and range of g.

Solution

Example 3

The graphs of f and g are given.



a. State the values of $f(1)$ and $g(1)$.

b. Which is larger, $f(-2)$ or $g(-2)$?

c. On what interval(s) is $g(x)$ increasing?

d. On what interval(s) is $f(x) \geq g(x)$?

e. State the solution to the equation $g(x) = 0$

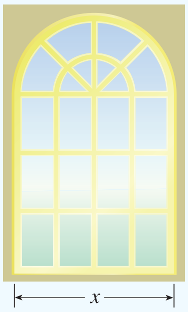
f. State the domain and range of f.

g. State the domain and range of g.

Solution

Example 4: 1.1.72

A Norman window has the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is 30 ft, express the area A of the window as a function of the width x of the window.



Solution

Example 5: 1.1.69

An open rectangular box with with volume $2\,m^3$ has a square base. Express the surface area of the box as a function of the length of a side of the base.

Solution

Example 6: 1.1.33

Let

$$f(x) = 3x^2 - x + 2$$

Compute the following and simplify your answer

a. $f(2)$

b. $f(-5)$

c. $f(a)$

d. $2f(a)$

e. $f(2a)$

Solution

Example 7

Let

$$g(x) = \sqrt{9x^2 + 4}$$

Compute the following and simplify your answer

a. $g(0)$

b. $g(a^2)$

c. $[g(a)]^2$

d. $g(a+h)$

e. $g(5-a)$

Solution

Difference quotients are a measure of the average rate of change of a functions over an interval. These will be treated in more detail when we discuss the tangent line to a curve. For now, consider this as 'practicing' your algebra skills.

Example 8

Suppose that $f(x) = 2x - 5$ and $h \neq 0$, evaluate the difference quotient

$$\frac{f(a+h) - f(a)}{h}$$

and simplify your answer.

Solution

Note: In this set of examples, domains for simple functions will be shown. Advanced techniques to determine the domain of more complicated functions will be covered in Lecture 2.

Example 9

Find the domain of the function, and report your answer using interval notation.

a. $f(x) = \dfrac{1}{x}$

b. $g(x) = \dfrac{2}{x-3}$

c. $h(x) = \dfrac{x-5}{x+10}$

d. $k(x) = \dfrac{x+4}{x^2-25}$

Solution

Example 10

Find the domain of the function, and report your answer using interval notation.

a. $f(x) = \sqrt{x}$

b. $g(x) = \sqrt{2 + x}$

c. $h(x) = \sqrt[4]{4 - x}$

d. $k(x) = \sqrt[6]{2x - 10}$

Solution

Example 11

Find the domain of the function, and report your answer using interval notation.

a. $f(x) = \sqrt[3]{x}$

b. $g(x) = \sqrt[5]{7 + x}$

c. $h(x) = \sqrt[9]{1 - 2x}$

Solution