1.2 A Catalog of Essential Functions

Functions are the central objects of study in calculus. What follows, is a survey of the functions that frequently arise in calculus and their applications: polynomials, rationals, exponential, logarithmic, and trigonometric.

When going through these notes, focus on the shape of the graphs, and the behaviours (ie. domain and range) for each type of function.

Polynomials

Many common functions are polynomial functions. This group of functions includes: the linear function, the quadratic function, and a small subclass of power functions. Here are some examples of polynomial functions:

$f(x) = 5x^{10} + 2x^7 - 4$ $\qquad k(t) = 54 + 2t$ $\qquad h(s) = s^{100} - 4s + 5$

$g(x) = \frac{3}{4}x^3 - \pi x^2 - \sqrt{2}x + 9$ $\qquad u(x) = 5$ $\qquad s(t) = \sqrt{5} - 3t^4$

> **Definition 1: Polynomial Functions**
>
> A **polynomial** of degree n is a function of the form
>
> $$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$
>
> where each of the $a's$ are real numbers (called the **coefficients** of the polynomial)

Polynomials are easily recognizable, because all of the exponents on the variable in the expression are non-negative and whole.

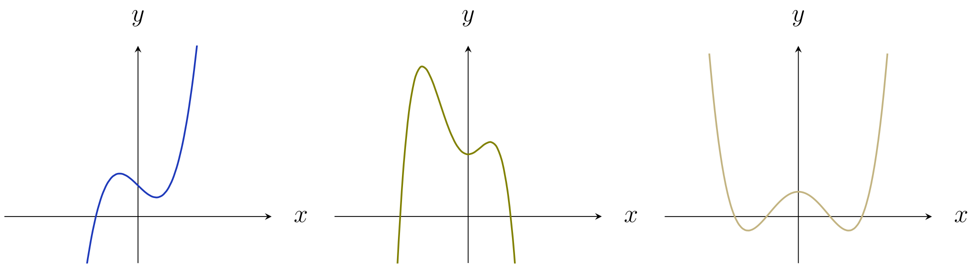
The **order** or degree of a polynomial is specified by the highest power in the expression.

$f(x) = \sqrt{2}x^5 + \pi x + 17$; order 5 polynomial

$f(x) = 2x + 1$; order 1 polynomial

$f(x) = 11$; order 0 polynomial

The graphs of polynomials are smooth and continuous (meaning that have no breaks, no jumps, no holes, and no gaps) in their graphs. In other words, the domain of a polynomial is $\mathbb{R}$.



Linear Functions

Linear functions are polynomials of degree 0 and 1. Linear functions have form $f(x) = a_1x + a_0$ but are more commonly represented using the **slope-intercept** form

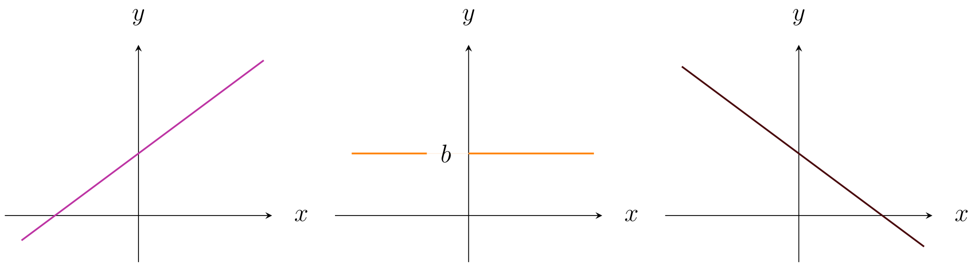
$$y = mx + b$$

where

m = slope of the line

b = y–intercept

Graphs of linear functions are lines, and their behaviour is controlled by the value the slope, m



$m > 0$	$m = 0$	$m < 0$
Domain: $\mathbb{R}$	Domain: $\mathbb{R}$	Domain: $\mathbb{R}$
Range: $\mathbb{R}$	Range: $y = b$	Range: $\mathbb{R}$

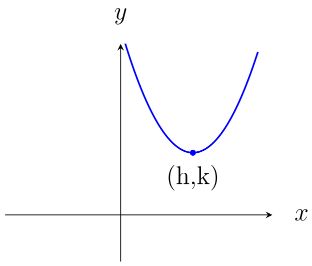
Quadratic Functions

Quadratic functions are polynomials of degree 2. The have form $f(x) = a_2x^2 + a_1x + a_0$ but are more commonly represented using the **vertex-form**

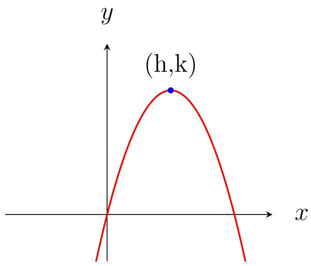
$$y = a(x - h)^2 + k$$

where (h, k) = coordinates of the vertex.

Graphs of quadratic functions are parabolas, and their behaviour is controlled by the value of the constant coefficient in front of the quadratic factor (ie. the number in front of the x^2)



$f(x) = a(x-h)^2 + k; \quad a > 0$
Domain: $\mathbb{R}$
Range: $[k, \infty)$



$f(x) = a(x-h)^2 + k; \quad a < 0$
Domain: $\mathbb{R}$
Range: $(-\infty, k]$

Exponential Functions

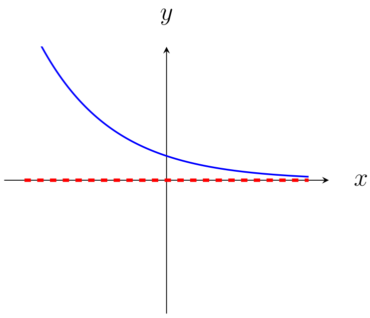
Another type of function which is frequently encountered in calculus are exponential functions.

Definition 2: Exponential Functions

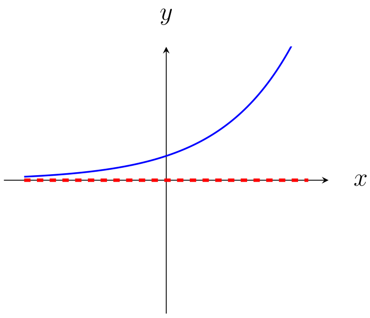
n **exponential function** is a function having form $f(x) = a^x$ where $a \in \mathbb{R}^{>0}$

- If $a > 1$ then the exponential function is classed as a **growth** function.
- If $0 < a < 1$ then the exponential function is classed as a **decay** function.

The graph of an exponential functions is controlled by the base, a, and has a horizontal asymptote at $y = 0$



$f(x) = a^x; \quad 0 < a < 1$
Domain: $\mathbb{R}$
Range: $(0, \infty)$



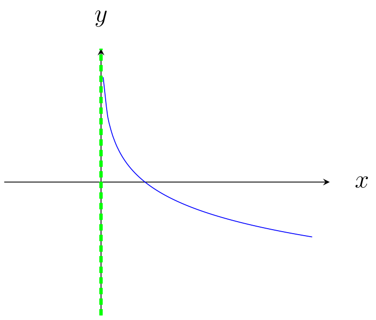
$f(x) = a^x; \quad a > 1$
Domain: $\mathbb{R}$
Range: $(0, \infty)$

Logarithmic Functions

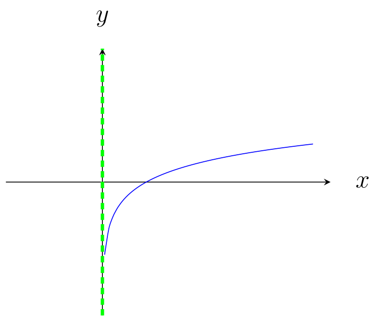
The logarithmic function is the inverse of the exponential function.

> **Definition 3: Logarithmic Functions**
>
> A **logarithmic function** is a function having form $f(x) = \log_a x$ where $a \in \mathbb{R}^{>0}$

The graph of a logarithmic function is controlled by its base, a, and has a vertical asymptote at $x = 0$



$f(x) = \log_a x; \quad 0 < a < 1$
Domain: $(0, \infty)$
Range: $\mathbb{R}$



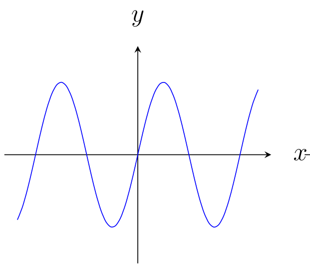
$f(x) = \log_a x; \quad a > 1$
Domain: $(0, \infty)$
Range: $\mathbb{R}$

Remark
Since the exponential and logarithmic functions are inverses of each other, observe that that their domains and range have are interchanged.

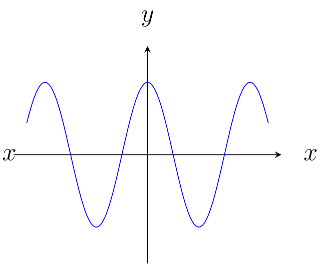
Trigonometric Functions

The last type of function we will review are the trigonometric functions. These are functions which relate the angle of a right-angle triangle to the ratio of two of its sides. There are six functions that are the core of trigonometry. There are three primary ones that you need to understand completely are: sine, cosine, and tangent.

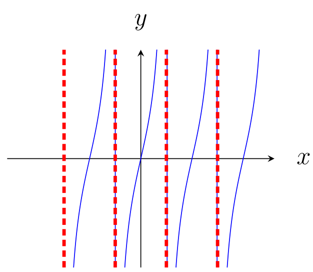
Here are the graphs for each of them:



$f(x) = \sin(x)$
Domain: $\mathbb{R}$
Range: $[-1, 1]$



$f(x) = \cos(x)$
Domain: $\mathbb{R}$
Range: $[-1, 1]$



$f(x) = \tan(x)$
Domain: $\mathbb{R} \backslash \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \ldots$
Range: $(-\infty, \infty)$