1.3 New Functions from Old Functions

Beginning with a small group of essential functions, we can create new functions from old ones using transformations, algebraic operations, and compositions.

Transformations

These are actions move and/or resize the graph of functions in the Cartesian plane. There are three types of actions: **translation, dilation,** and **reflection**.

Translation

This action involves moving the graph in space without changing its size, shape, or orientation. There are a total of four translations; two for vertical shifting, and two to horizontal shifting.

Result 1: Translations

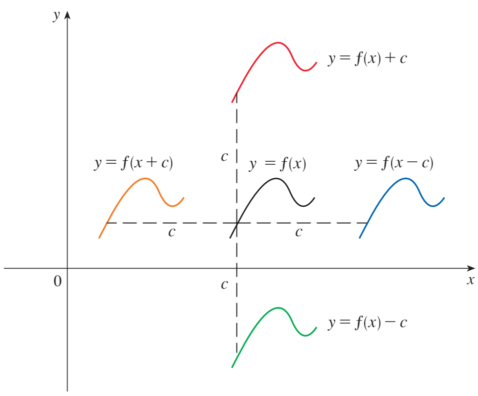
Suppose that $c > 0$. To obtain the graph of

$y = f(x) + c$; shift the graph of $f(x)$ **up** by c units

$y = f(x) - c$; shift the graph of $f(x)$ **down** by c units

$y = f(x + c)$; shift the graph of $f(x)$ **left** by c units

$y = f(x - c)$; shift the graph of $f(x)$ **right** by c units



Example 1

Describe the transformation that has been applied to $f(x)$ in order to obtain $g(x)$. Sketch the graph of $g(x)$ and state its domain and range.

a. $f(x) = x^2$ $\quad g(x) = x^2 - 3$

b. $f(x) = x^3$ $\quad g(x) = x^3 + 10$

c. $f(x) = x^4$ $\quad g(x) = (x-5)^4$

d. $f(x) = |x|$ $\quad g(x) = |x+8|$

Solution

Example 2

Describe the transformation(s) that have been applied to $f(x)$ in order to obtain $g(x)$. Sketch the graph of $g(x)$ and state its domain and range.

a. $f(x) = x^7$ $\quad g(x) = (x-4)^7 + 1$

b. $f(x) = \log x$ $\quad g(x) = \log(x+2) - 5$

c. $f(x) = e^x$ $\quad g(x) = e^{x-9}$

Solution

Dilation

Dilation involves the resizing r rescaling of the graph. It is typically described as the stretching or compressing of a graph.

Result 2: Dialation

Suppose that $c > 0$. To obtain the graph of

$y = cf(x)$; **vertically stretch** the graph of $f(x)$ by c units

$y = \frac{1}{c}f(x)$; **vertically compress** the graph of $f(x)$ by c units

$y = f(cx)$; **horizontally compress** the graph of $f(x)$ by c units

$y = f\left(\frac{1}{c}x\right)$; **horizontally stretch** the graph of $f(x)$ by c units

Reflection

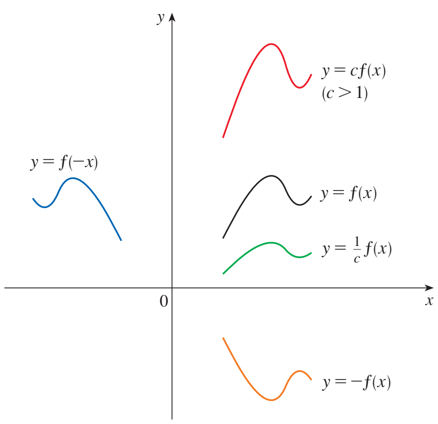
Reflection involves taking the entire graph and flipping it across either the y−axis, or the x−axis.

Result 3: Reflections

Suppose that $c > 0$. To obtain the graph of

$y = -f(x)$; reflect the graph of $y = f(x)$ about the x−axis

$y = f(-x)$; reflect the graph of $y = f(x)$ about the y−axis



Example 3

Describe the transformation that has been applied to $f(x)$ in order to obtain $g(x)$. Sketch the graph of $g(x)$.

a. $f(x) = x^4$ $\qquad g(x) = 2x^4$

b. $f(x) = x^5$ $\qquad g(x) = \frac{1}{3}x^5$

c. $f(x) = \cos(x)$ $\qquad g(x) = \cos(2x)$

d. $f(x) = |x|$ $\qquad g(x) = \left|\frac{1}{4}x\right|$

e. $f(x) = e^x$ $\qquad g(x) = -e^x$

f. $f(x) = \ln(x)$ $\qquad g(x) = \ln(-x)$

Solution

Example 4: 1.3.10

Graph the function by hand, not by plotting points, but by starting with the graph of one of the standard functions and then applying the appropriate transformations. State the domain and range of the function.

$$y = (x + 1)^2$$

Solution

Example 5: 1.3.12

Graph the function by hand, not by plotting points, but by starting with the graph of one of the standard functions and then applying the appropriate transformations. State the domain and range of the function.

$$y = 1 - x^3$$

Solution

Example 6: 1.3.14

Graph the function by hand, not by plotting points, but by starting with the graph of one of the standard functions and then applying the appropriate transformations. State the domain and range of the function.

$$y = -\sqrt{x} - 1$$

Solution

Example 7: 1.3.16

Graph the function by hand, not by plotting points, but by starting with the graph of one of the standard functions and then applying the appropriate transformations. State the domain and range of the function.

$$y = 1 + \frac{1}{x^2}$$

Solution

Example 8: 1.3.18

Graph the function by hand, not by plotting points, but by starting with the graph of one of the standard functions and then applying the appropriate transformations. State the domain and range of the function.

$$y = -(x-1)^2 + 3$$

Solution

Example 9: 1.3.22

Graph the function by hand, not by plotting points, but by starting with the graph of one of the standard functions and then applying the appropriate transformations. State the domain and range of the function.

$$y = 2 - 2\cos(x)$$

Solution

Example 10: 1.3.26

Graph the function by hand, not by plotting points, but by starting with the graph of one of the standard functions and then applying the appropriate transformations. State the domain and range of the function.

$$y = \left|\sqrt{x} - 1\right|$$

Solution

Algebraic Operations

Another way to create new functions with old functions, is by combining them with algebraic operations (ie. $+, -, \times, \div$).

Theorem 1

Suppose that $f(x)$ and $g(x)$ are two functions with domains D_f and D_g respectively. Then the operation of addition, subtraction, multiplication, division, along with the resultant domain are defined as follows:

i. $(f+g)(x) = f(x) + g(x)$; $D_{f+g} = D_f \cap D_g$

ii. $(f-g)(x) = f(x) - g(x)$; $D_{f-g} = D_f \cap D_g$

iii. $(fg)(x) = f(x) \times g(x)$; $D_{f\times g} = D_f \cap D_g$

iv. $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$; $D_{f/g} = D_f \cap D_g$; $g(x) \neq 0$

Example 11: 1.3.34

Let

$$f(x) = \frac{1}{x-1} \quad \text{and} \quad g(x) = \frac{1}{x} - 2$$

Determine each of the following. Simplify your solutions and state their domains.

a. $f + g$

b. $f - g$

c. $f \times g$

d. $\frac{f}{g}$

Solution

Composition of Functions

Composition is another way of combining several functions to create a new ones. This is where two or more functions are combined in a manner such that one becomes the input for the other.

Definition 1: Composition

Suppose that $f(x)$ and $g(x)$ are two functions with domains D_f and D_g respectively. Then the **composition** of f and g, denoted as $f \circ g$ is defined to be

$$(f \circ g)(x) = f(g(x))$$

The domain of $D_{(f \circ g)(x)} = D_{f(g(x))} \cap D_g$

Remark
Order matters in a composition of functions. In general $(f \circ g)(x) \neq (g \circ f)(x)$

Example 12: 1.3.55

Use the table below to evaluate the expression

x	1	2	3	4	5	6
$f(x)$	3	1	5	6	2	4
$g(x)$	5	3	4	1	3	2

a. $f(g(3))$

b. $g(f(2))$

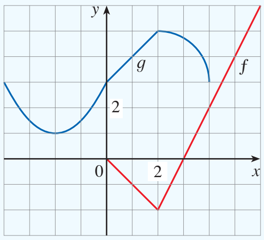
c. $(f \circ g)(5)$

d. $(g \circ f)(5)$

Solution

Example 13: 1.3.57

Use the given graphs of f and g to evaluate each expression, or explain why it is undefined.



a. $f(g(2))$

b. $g(f(0))$

c. $(f \circ g)(0)$

d. $(g \circ f)(6)$

e. $(g \circ g)(-2)$

f. $(f \circ f)(4)$

Example 14: 1.3.36

Let

$$f(x) = \frac{1}{x} \qquad \text{and} \qquad g(x) = 2x + 1$$

Determine the following compositions and their domains

a. $f \circ g$

b. $g \circ f$

Solution

Example 15: 1.3.40

Let

$$f(x) = \sqrt{5 - x} \qquad \text{and} \qquad g(x) = \sqrt{x - 1}$$

Determine the following compositions and their domains

a. $f \circ f$

b. $g \circ g$

Solution

Example 16: 1.3.42

Find $f \circ g \circ h$ if

$$f(x) = |x - 4| \qquad g(x) = 2^x \qquad h(x) = \sqrt{x}$$

Solution

Example 17: 1.3.44

Find $f \circ g \circ h$ if

$$f(x) = \tan x \qquad g(x) = \frac{x}{x-1} \qquad h(x) = \sqrt[3]{x}$$

Solution

Example 18: 1.3.46

Express the function in the form $f \circ g$ if

$$F(x) = \cos^2 x$$

Solution

Example 19: 1.3.48

Express the function in the form $f \circ g$ if

$$G(x) = \sqrt[3]{\frac{x}{1+x}}$$

Solution

Example 20: 1.3.52

Express the function in the form $f \circ g \circ h$ if

$$H(x) = \sqrt[8]{2+|x|}$$

Solution

Example 21: 1.3.54

Express the function in the form $f \circ g \circ h$ if

$$H(t) = \cos(\sqrt{\tan t} + 1)$$

Solution