

1.4 Exponential Functions

Exponential functions are used in a wide variety of scientific and non-scientific applications (eg. population growth, radioactive decay, compound interest, ...).

Definition 1: Exponential Function

An **exponential function** is a function that has form

$$f(x) = a^x$$

where $a \in \mathbb{R}^{>0}$; $x \in \mathbb{Z}$

There are three scenarios for x .

1. $x = n$ and $n \in \mathbb{Z}^{>0} \Rightarrow a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n\text{-times}}$
2. $x = -n$ and $n \in \mathbb{Z}^{>0} \Rightarrow a^{-n} = \frac{1}{a^n} = \frac{1}{\underbrace{a \cdot a \cdot a \cdots a}_{n\text{-times}}}$
3. $x = 0 \Rightarrow a^0 = 1$

The following rules/laws specify how to work with exponents. These will be useful for simplifying expressions.

Result 1: Law of Exponents

If $a, b \in \mathbb{R}^{>0}$ and $x, y \in \mathbb{R}$ then

1. $a^x a^y = a^{x+y}$
2. $\frac{a^x}{a^y} = a^{x-y}$
3. $(a^x)^y = a^{xy}$
4. $(ab)^x = a^x b^x$

Example 1: 1.4.2

Use the Law of Exponents to rewrite and simplify each expression

a. $2x^2 (3x^5)^2$

b. $(2x^{-2})^{-3} x^{-3}$

c. $\frac{3a^{3/2} \cdot a^{1/2}}{a^{-1}}$

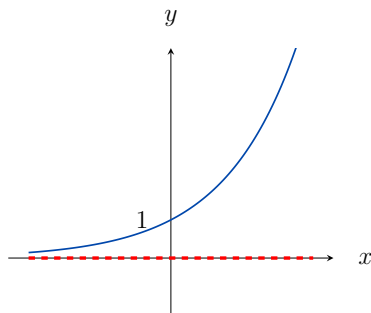
d. $\frac{\sqrt{a}\sqrt{b}}{\sqrt[3]{ab}}$

Solution

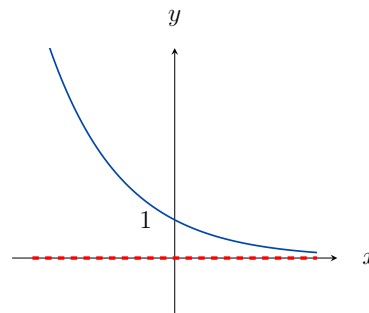
Graphs of Exponential Functions

The value of a , controls the behaviour of the function.

- If $a > 1$, then $f(x) = a^x$ is an increasing function.
- If $0 < a < 1$, then $f(x) = a^x$ is a decreasing function.



$$f(x) = a^x; \quad a > 1$$

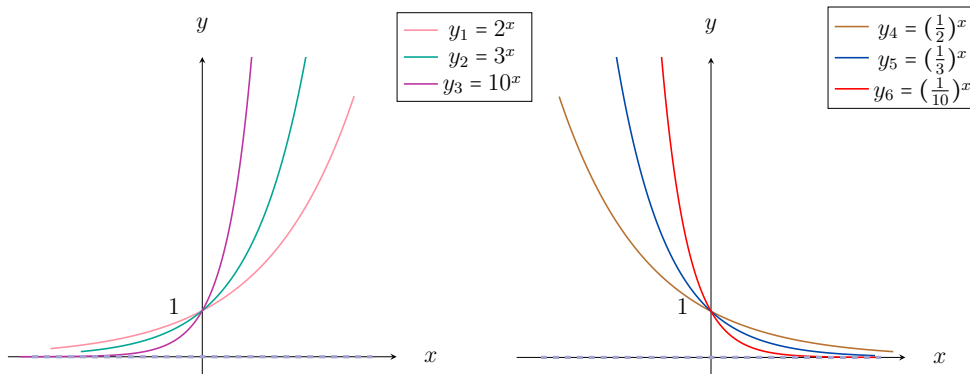


$$f(x) = a^x; \quad 0 < a < 1$$

Remark

There is a horizontal asymptote at $y = 0$; and the function passes through $(0, 1) \because f(0) = a^0 = 1$

The magnitude of a , determines how steep the curve is. For $a > 1$, the larger the value of a , the steeper the curve. For $0 < a < 1$ the opposite is true.



Example 2

Starting with the graph of $y = 5^x$, write an equation of the graph that results from

- shifting 3 units downwards
- shifting 4 units left
- reflecting it into the x -axis and then shifting it up 2 units
- shifting it 7 units to the right, reflecting it into the y -axis, and shifting it up by 1

Solution

Example 3: 1.4.10

Make a rough sketch by hand of the given function. State the domain and range.

$$h(x) = 2\left(\frac{1}{2}\right)^x - 3$$

Solution**Example 4: 1.4.12**

Make a rough sketch by hand of the given function. State the domain and range.

$$y = 4^{x+2}$$

Solution**Example 5: 1.4.14**

Make a rough sketch by hand of the given function. State the domain and range.

$$y = e^{|x|}$$

Solution

Example 6: 1.4.18

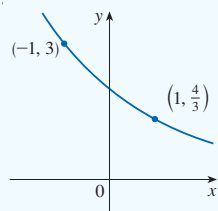
Find the domain of the function

a. $g(t) = \sqrt{10^t - 100}$

b. $g(t) = \sin(e^t - 1)$

Solution**Example 7: 1.4.20**

Find the exponential function $f(x) = Cb^x$ whose graph is given.

**Solution**

Example 8: 1.4.21

If $f(x) = 5^x$ show that

$$\frac{f(x+h) - f(x)}{h} = 5^x \left(\frac{5^h - 1}{h} \right)$$

Solution**Example 9: 1.4.22**

Suppose you are offered a job that lasts one month. Which of the following methods of payment do you prefer?

- i. One million dollars at the end of the month.
- ii. One cent on the first day of the month, two cents on the second day, four cents on the third day, and, in general, 2^{n-1} cents on the n^{th} day.

Solution

Example 10: 1.4.30

A gray squirrel population was introduced in a certain region 18 years ago. Biologists observe that the population doubles every six years, and now the population is 600.

- a. What was the initial squirrel population?
- b. What is the expected squirrel population t years after introduction?
- c. Estimate the expected squirrel population 10 years from now.

Solution