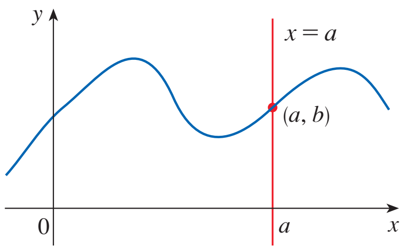
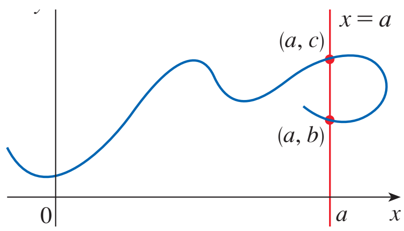
1.5 Inverse Functions and Logarithms

Inverse Functions

Given the graph of a curve, a relation qualifies as a **function** iff it passes the **vertical line test**.

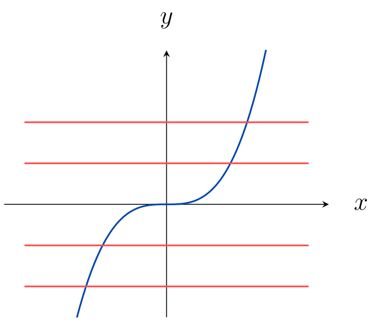


Function

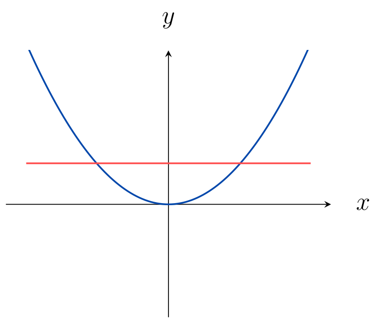


Not a function

A function can be classed as either one-to-one or not. A **one-to-one function**, passes the horizontal line test.



One-to-one



Not one-to-one

> **Definition 1: One-to-One Functions**
>
> A function is **one-to-one** iff $\forall\, x_1, x_2 \in D_f$
>
> $$f(x_1) \neq f(x_2) \qquad \text{whenever } x_1 \neq x_2$$

If $f(x)$ is a one-to-one function, then it has an **inverse**, $f^{-1}(x)$, (ie. given some value of y, it is possible to map it back to the unique value of x which generated it).

Definition 2: Inverse Functions

Let $f(x)$ be a one-to-one function with domain D_f and range R_f. The **inverse function**, f^{-1} has domain R_f and range D_f and is defined by

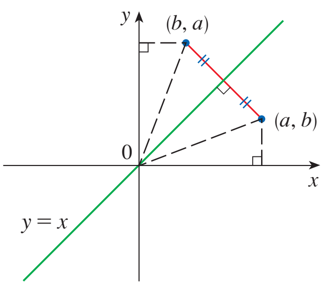
$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y \quad \forall y \in R_f$$

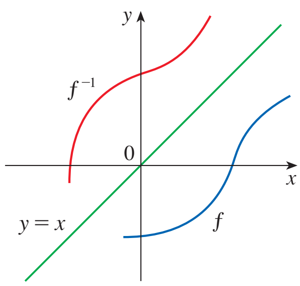
Result 1: Domain and Range of $f(x)$ and $f^{-1}(x)$

Let $f(x)$ be a one-to-one function with domain D_f and range R_f. Then

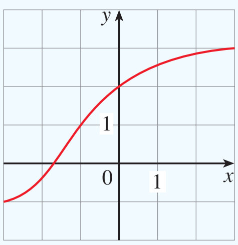
- the domain of $f^{-1}(x)$ is $D_{f^{-1}(x)} = R_f$
- the range of $f^{-1}(x)$ is $R_{f^{-1}(x)} = D_f$

Graphically, $f(x)$ and $f^{-1}(x)$ are reflections of each other along the line $y = x$.





Example 1



Consider the graph of $f(x)$ above.

a. What is the domain of $f(x)$?

b. What is the range of $f(x)$?

c. What is the domain of $f^{-1}(x)$?

d. What is the range of $f^{-1}(x)$?

e. What is the value of $f^{-1}(2)$ and $f(2)$?

f. What is the value of $f^{-1}(-1)$ and $f(-1)$?

g. What is the value of $f^{-1}(3)$?

Solution

Computing the Inverse of a Function

In many situations, we want to determine the inverse of a function starting from its algebraic form.

Result 2: Computing the Inverse of a Function

Suppose that $f(x)$ is a one-to-one function

1. Let $y = f(x)$
2. Interchange the y with the x's
3. Isolate for y. The resulting expression will be $f^{-1}(x)$.

Theorem 1

If $f(x)$ is a one-to-one function with domain D_f and range R_f, and $f^{-1}(x)$ is its inverse, then

$$\begin{aligned} (f^{-1} \circ f)(x) &= f^{-1}(f(x)) = x \qquad \forall x \in D_f \\ (f \circ f^{-1})(x) &= f(f^{-1}(x)) = x \qquad \forall x \in R_f \end{aligned}$$

Remark
In other words, the composition of f and its inverse is x

Example 2: 1.5.24

Find a formula for the inverse

$$g(x) = x^2 - 2x \qquad ; \qquad x \geq 1$$

Solution

Example 3: 1.5.26

Find a formula for the inverse

$$h(x) = \frac{6 - 3x}{5x + 7}$$

Solution

Example 4: 1.5.28

Find a formula for the inverse

$$y = 3\ln(x - 2)$$

Solution

Example 5: 1.5.30

Find a formula for the inverse

$$y = \frac{1 - e^{-x}}{1 + e^{-x}}$$

Solution

Example 6

Let $f(x) = 1 + \sqrt{x+3}$

a. Find the inverse of the function.

b. What is the domain of $f(x)$? What is the range of $f^{-1}(x)$?

c. What is the domain of $f^{-1}(x)$ and the range of $f(x)$?

d. Sketch a graph of $f(x)$ and $f^{-1}(x)$ on the same axis.

Solution

Logarithmic Function

Exponential functions are one-to-one functions. Therefore, they have an inverse. The inverse of an exponential function is a logarithmic functions, and vice versa.

Remark
This means that every exponential function can be turned into a logarithmic function, and vice versa.

Definition 3: Logarithmic Function

A **logarithmic function** is a function with form

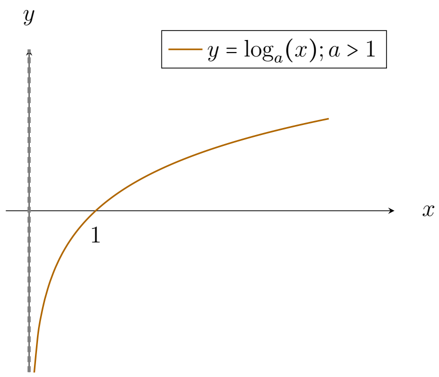
$$f(x) = \log_a x \qquad ; \qquad x > 0 \quad , \quad a \in \mathbb{R}^{>0}$$

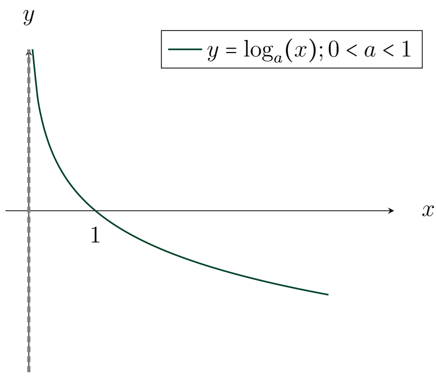
a is called the **base**.

Remark
$\log_{10} x$ is abbreviated to $\log x$ and $\log_e x$ is written as $\ln x$

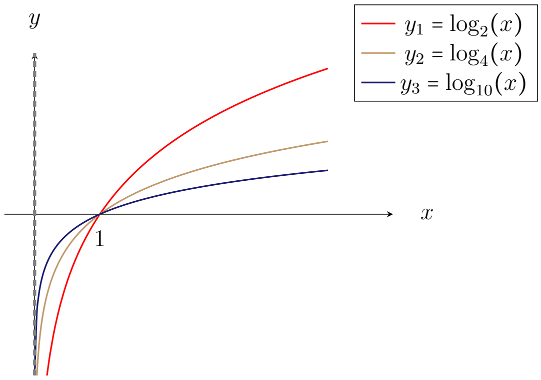
Graphs of Logarithmic Functions

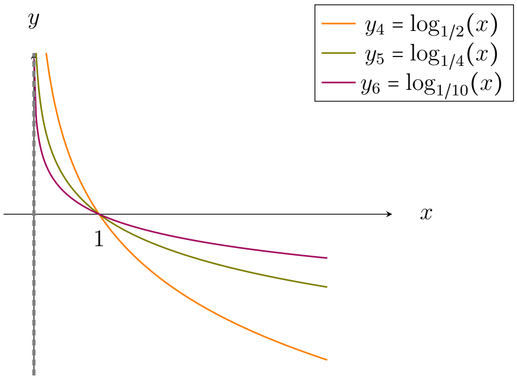
Logarithmic functions behave in a manner similar to that of exponential functions. When $a > 1$, the function is increasing, and when $0 < a < 1$ the function is decreasing.



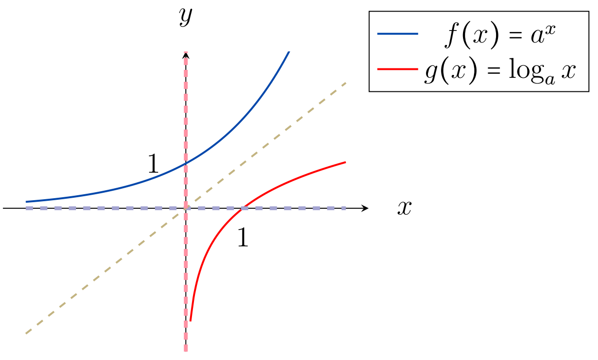


The magnitude of the base controls the steepness of the curve. When $a > 1$, the smaller the value of a, the faster it increases; and the larger the value of a, the slower it grows. The opposite is true when $0 < a < 1$.

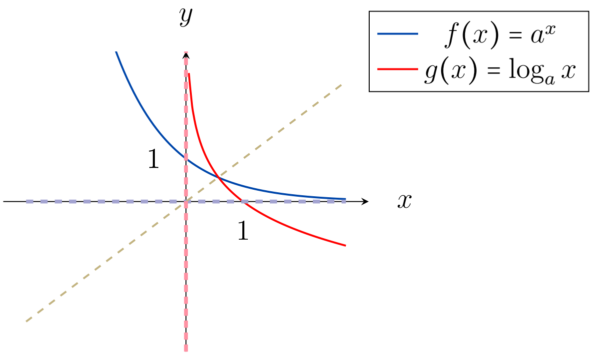




Graphs illustrating that logarithmic and exponential functions as inverses of each other.



$$a > 1$$
$$D_f = R_g = \mathbb{R}$$
$$R_f = D_g = (0, \infty)$$



$$0 < a < 1$$
$$D_f = R_g = \mathbb{R}$$
$$R_f = D_g = (0, \infty)$$

Example 7: 1.5.56

Sketch the function, and state the domain and range.

$$f(x) = \ln(x - 1) - 1$$

Solution

Result 3: Properties of Logarithms

Let $x, y \in \mathbb{R}^{>0}$. Then the following holds:

1. $\log_a(xy) = \log_a x + \log_a y$
2. $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
3. $\log_a(x^r) = r\log_a x \quad ; \quad$ where $r \in \mathbb{R}$
4. $\log_a x = \log_a y \quad \leftrightarrow \quad x = y$
5. $\log_a 1 = 0$
6. $\log_a a = 1$
7. $\log_a a^x = x \quad ; \quad$ can be generalized to $\log_a a^{f(x)} = f(x)$
8. $a^{\log_a x} = x \quad ; \quad$ can be generalized to $a^{\log_a f(x)} = f(x)$

Example 8

Find the exact value of each expression

a. $\log_3 81$

b. $\log_9 3$

c. $\ln \frac{1}{e^2}$

d. $\log_2 30 - \log_2 15$

e. $2\log_5 100 - 4\log_5 50$

f. $e^{3\ln 2}$

g. $e^{-2\ln 5}$

h. $e^{\ln(\ln e^3)}$

Solution

Example 9: 1.5.44

Use the properties of logarithms to expand each expression:

a. $\ln\sqrt{\frac{3x}{x-3}}$

b. $\log_2\left[(x^3+1)\sqrt[3]{(x-3)^2}\right]$

Solution

Example 10: 1.5.46

Express as a single logarithm

a. $3\ln(x-2)-\ln\left(x^2-5x+6\right)+2\ln(x-3)$

b. $c\log_a x - d\log_a y + \log_a z$

Solution

Example 11: 1.5.58

Solve the equation for x and leave your answer as an exact solution.

a. $\log_2\left(x^2 - x - 1\right) = 2$

b. $1 + e^{4x+1} = 20$

Solution

Example 12: 1.5.60

Solve the equation for x and leave your answer as an exact solution.

a. $\ln(\ln x) = 0$

b. $\frac{60}{1+e^{-x}} = 4$

Solution

Example 13: 1.5.60

Solve the inequality for x and leave your answer as an exact solution.

a. $1 < e^{3x-1} < 2$

b. $1 - 2\ln x < 3$

Solution

Example 14

Starting with the graph of $y = \log_5 x$, write an equation of the graph that results from

a. shifting 10 units downwards

b. shifting 8 units right

c. reflecting it into the $x-$axis and then shifting it down 2 units

d. shifting it 4 units to the left, reflecting it into the $y-$axis, and shifting it up by 3

Solution

Example 15

Sketch the graph of $g(x)$ and state its domain and range.

$$g(x) = \log_2(x - 3)$$

Solution

Example 16

Sketch the graph of $g(x)$ and state its domain and range.

$$g(x) = \log_3(x + 4) + 1$$

Solution

Example 17

Sketch the graph of $g(x)$ and state its domain and range.

$$g(x) = \log_{\frac{1}{10}}(2 - x) - 3$$

Solution

Example 18

Compute the domain for each of the following functions

a. $f(x) = \ln(x - 3)$

b. $g(x) = \log(x + 5) + 2$

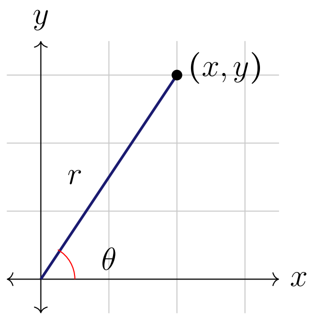
c. $h(x) = \log_5(2 - x) - 3$

d. $k(x) = \ln(e^x - 6)$

Solution

Trigonometric Functions

Let r be the distance from the origin to the point (x, y) and let θ be the angle on the terminal side of the ray.

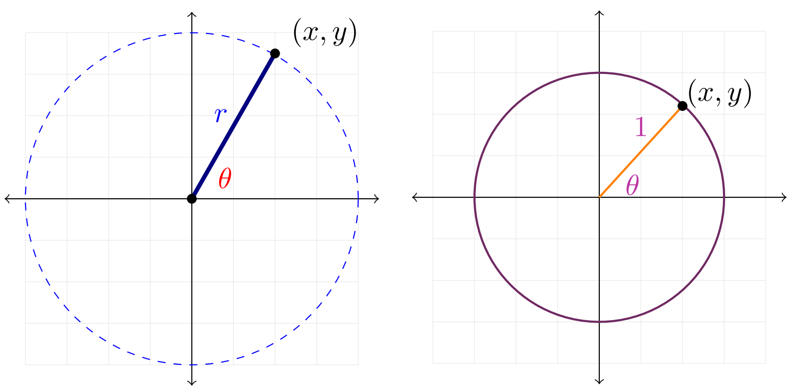


Then the six trigonometric functions are define the ratio of distances from the origin to the point (x, y)

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

$$\csc\theta = \frac{r}{y} \qquad \sec\theta = \frac{r}{x} \qquad \cot\theta = \frac{x}{y}$$

If θ is allowed to take on values from 0° to 360° then the path traced by r, is a circle with radius r centred at $(0, 0)$.



If $r = 1$ then this will generate the unit circle, and the six trigonometric functions can be revised to

$$\sin\theta = y \qquad \cos\theta = x \qquad \tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$$

$$\csc\theta = \frac{1}{y} = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{x} = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{x}{y} = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

In calculus all angles are measured in radians. The table below shows some of the most common values for $\sin\theta$ and $\cos\theta$.

θ	0	$\frac{\pi}{6} = 30°$	$\frac{\pi}{4} = 45°$	$\frac{\pi}{3} = 60°$	$\frac{\pi}{2} = 90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

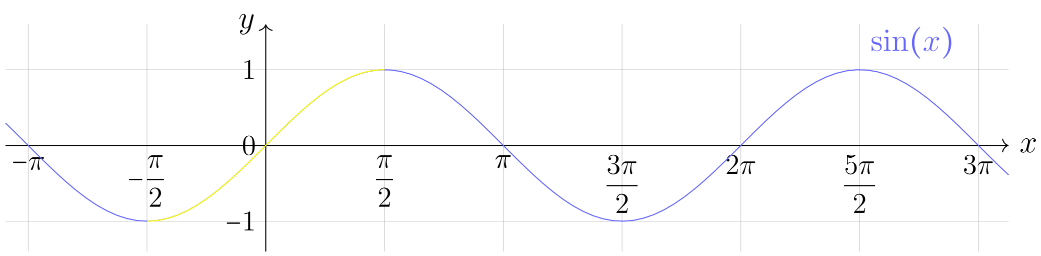
Inverse Trigonometric Functions

Trigonometric functions are not one-to-one but we can make them so by restricting their domains. The key to is to restrict the values of x so that all values of y are accounted for.

There are six inverse trigonometric functions. For the purpose of the course, you need to be familiar with the inverse of sine, cosine, and tangent.

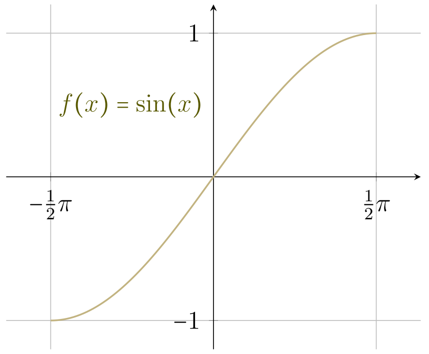
Sine Inverse

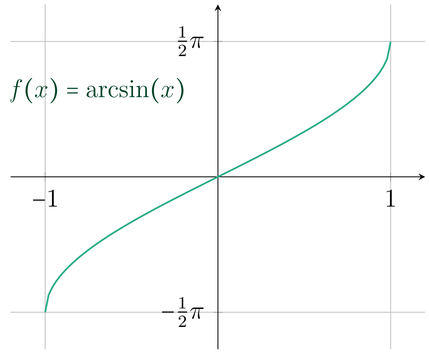
Here is the graph of $\sin x$.



By convention we restrict the domain of the sine to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ where it is one-to-one, and call its inverse on this restricted domain the arcsine function or the inverse of sine function.

On the left, the graph of sine with the restricted domain is shown. On the right the we have the graph of arcsine.





Definition 4: Arcsine Function

The inverse of the sine function is called **arcsine** and is defined as follows

$$y = \arcsin(x) \quad \Leftrightarrow \quad \sin y = x \quad ; \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

Remark
$\arcsin(x)$ is also denoted as $\sin^{-1}(x)$

Result 4: Composition of Sine and Sine Inverse

$$\sin^{-1}(\sin x) = x \quad ; \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$
$$\sin(\arcsin x) = x \quad ; \quad -1 \leq x \leq 1$$

Example 19

Compute the following values.

a. $\arcsin(0)$

b. $\arcsin(-1)$

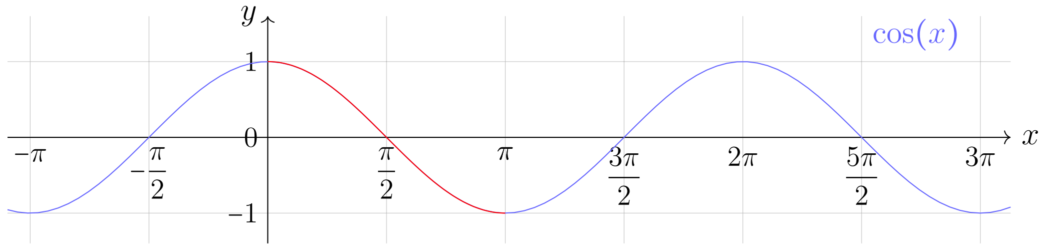
c. $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

d. $\sin^{-1}\left(\frac{-\sqrt{2}}{2}\right)$

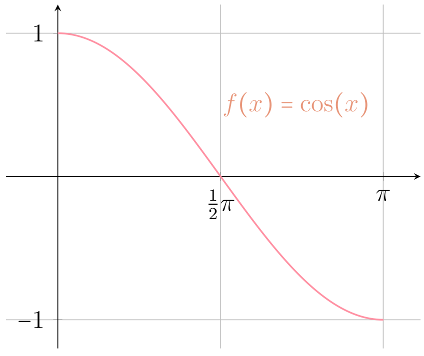
Solution

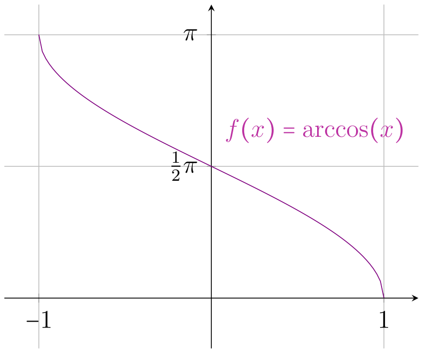
Cosine Inverse

Below is the graph of the cosine function.



The graph of $\cos x$ with the restricted domain is shown on the left, and the inverse of cos is on the right.





Definition 5: Arccosine Function

The inverse of the cosine function is called **arccosine** and is defined as follows

$$y = \arccos(x) \quad \Leftrightarrow \quad \cos y = x \quad ; \quad 0 \le y \le \pi$$

Remark
$\arccos(x)$ is also denoted as $\cos^{-1}(x)$

Result 5: Composition of Cosine and Cosine Inverse

$$\cos^{-1}(\cos x) = x \quad ; \quad 0 \le x \le \pi$$

$$\cos(\arccos x) = x \quad ; \quad -1 \le x \le 1$$

Example 20

Compute the following values.

a. $\arccos(1)$

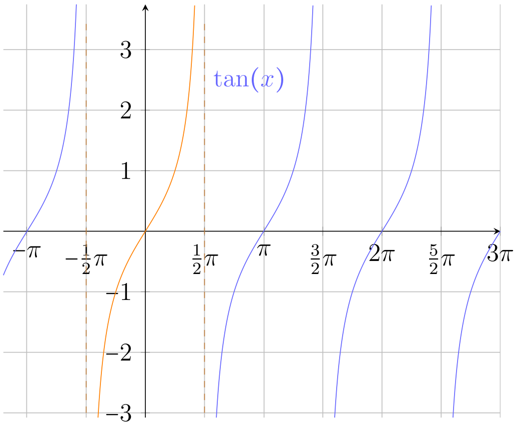
b. $\arccos(0)$

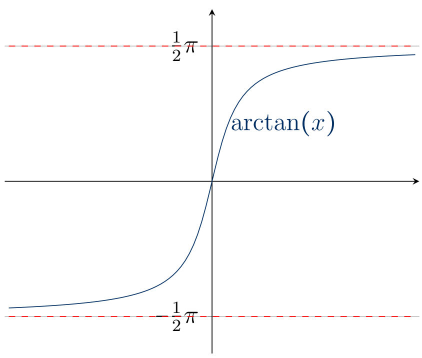
c. $\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)$

Solution

Tangent Inverse

As with the other two functions, we restrict the domain of tangent to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and call its inverse on this restricted domain the arctangent function or the inverse of tangent function.





Definition 6: Arctangent Function

The inverse of the tangent function is called **arctangent** and is defined as follows

$$y = \arctan(x) \quad \Leftrightarrow \quad \tan y = x \quad ; \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

Remark
$\arctan(x)$ is also denoted as $\tan^{-1}(x)$

Result 6: Composition of Tangent and Tangent Inverse

$$\tan^{-1}(\tan x) = x \quad ; \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$
$$\tan(\arctan x) = x \quad ; \quad -\infty < x < \infty$$

Example 21

Compute the following values.

a. $\arctan(0)$

b. $\arctan(1)$

c. $\tan^{-1}(-1)$

d. $\tan^{-1}\left[\tan\left(\frac{3\pi}{4}\right)\right]$

Solution

Example 22

Compute the following values.

a. $\sec^{-1}(\sqrt{2})$

b. $\arctan(\sqrt{3})$

c. $\tan(\arctan(10))$

d. $\sin^{-1}\left[\sin\left(\frac{7\pi}{3}\right)\right]$

e. $\cot^{-1}(1)$

Solution

Example 23

Determine the exact value of the expression

a. $\sin\left[\arctan\left(\frac{3}{4}\right)\right]$

b. $\sec\left[\arcsin\left(\frac{5}{13}\right)\right]$

c. $\tan(\sin^{-1} x)$

Solution