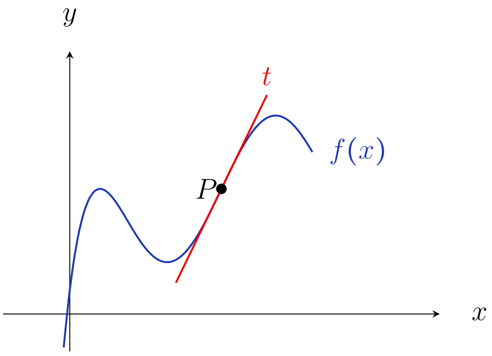
Lecture 8: The Tangent and Velocity Problems

The word **tangent** is derived from the Latin word *tangens* which means "touching". In mathematics, a **tangent line** to a curve is a straight line that touches the curve of a function at **one and only one point**.



Tangent lines are important for two reasons:

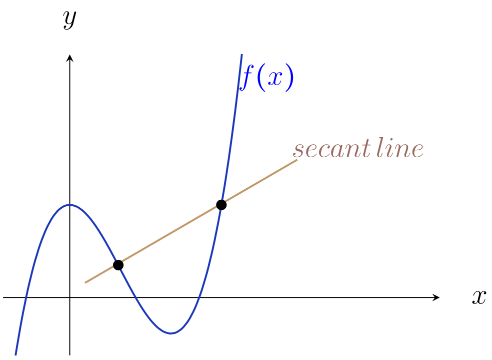
1. They allow us to find the slope of of a curved function at any point on the curve;

2. the slopes of tangent lines forms the fundamental interpretation for the derivative of a function - a real valued quantity which measures how sensitive a function is, to change with respect to its argument

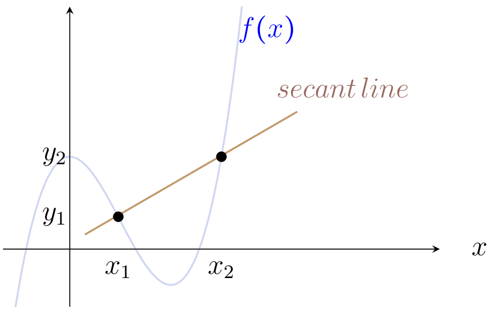
Secant Lines

Definition 1: Secant Line

A **secant line** is a straight line that joins two points on a function.



Let (x_1, y_1) and (x_2, y_2) be two points on a function. We can join the two points with a secant line.



> **Remark**
> The **slope of the secant line** corresponds to the **average rate of change** of the function.

Result 1: Average Rate of Change

Let $f(x)$ be a function containing the points (x_1, y_1) and (x_2, y_2). Then

$$\text{average rate of change } = m_{secant} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Equation of the Secant Line

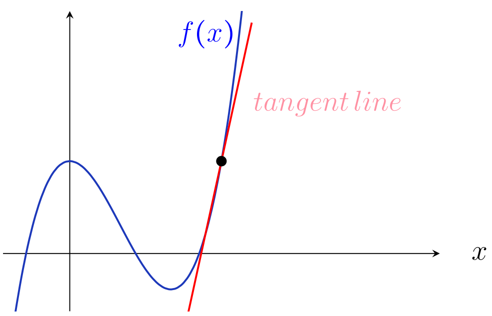
The equation of the secant line can be written in two ways:

$$y = m_{sec}\, x + b \qquad ; \qquad \text{slope - intercept form}$$
$$y - y_1 = m_{sec}\,(x - x_1) \qquad ; \qquad \text{point - slope form}$$

Tangent Lines

Definition 2: Tangent Line

A **tangent line** is a straight line that touches a function at one and only one point.



> **Remark**
> The **slope of the tangent line** corresponds to the **instantaneous rate of change** of the function.

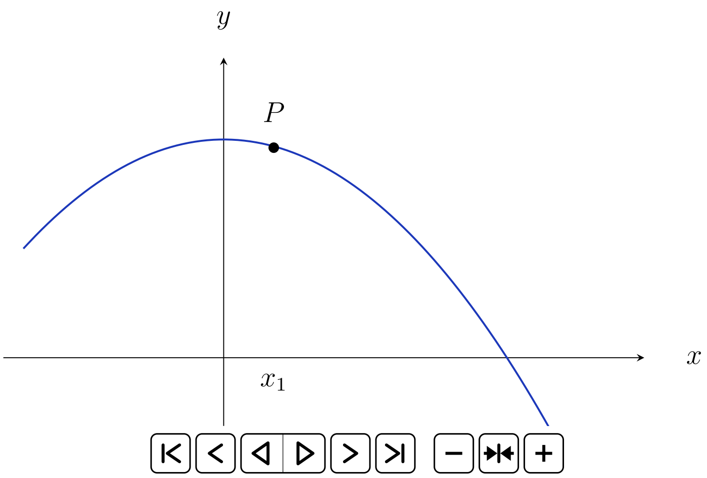
In many situations, we want to determine the equation of the tangent line given some point on the function. However, in order to do so, we need two points on the curve to calculate the slope of the line, but only have one.

To overcome this issue, consider the following proposition: use the slope of the secant line to approximate the slope of the tangent line.

Using the Slope of the Secant Line to Approximate the Slope of the Tangent Line

Let P be a fixed point on the function $f(x)$ for which we would like to the tangent line drawn at.

Next, pick a point on the curve near P and call it Q. Now that we have two points on the curve, connect P and Q with a secant line, and compute the slope. Once the slope has been calculated, move Q closer to P, and repeat.



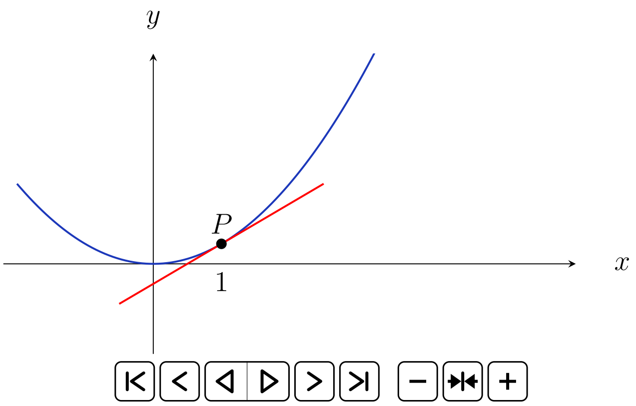
The reason why this work is that as Q moves closer to P along the curve, the secant line rotates along P and approaches the tangent line. What this shows is that the slope of the tangent line can be viewed as the limit of the slopes of the secant lines as x_2 approaches x_1 . As a result,

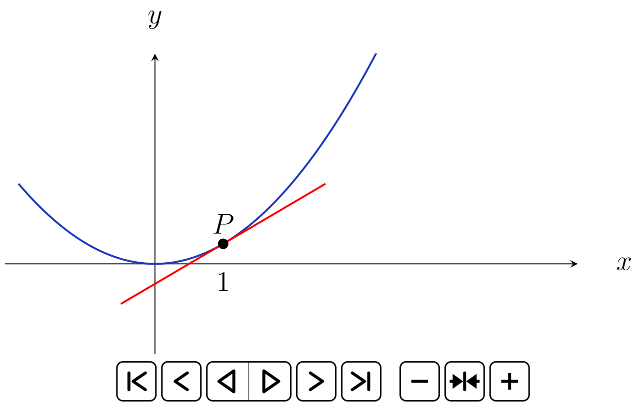
$$\lim_{Q \to P} m_{PQ} = m_{tangent}$$

Example 1

Find the equation of the tangent line of $f(x) = x^2$ at the point $P = (1, 1)$.

Solution





Example 2: 2.1.4

The point $P(0.5, 0)$ lies on the curve $y = \cos \pi x$.

a. If Q is the point $(x, \cos \pi x)$, find the slope of the secant line PQ (correct to six decimal places) for the following values of x :

   i. 0
   ii. 0.4
   iii. 0.49
   iv. 0.499
   v. 1
   vi. 0.6
   vii. 0.51
   viii. 0.501

b. Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(0.5, 0)$.

c. Using the slope from part (b), find an equation of the tangent line to the curve at $P(0.5, 0)$.

d. Sketch the curve, two of the secant lines, and the tangent line.

Solution

Example 3: 2.1.6

If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height in meters t seconds later is given by

$$y = 10t - 1.86t^2$$

a. Find the average velocity over the given time intervals:

   i. $[1, 2]$
   ii. $[1, 1.5]$
   iii. $[1, 1.1]$
   iv. $[1, 1.01]$
   v. $[1, 1.001]$

b. Estimate the instantaneous velocity when $t = 1$.

Solution

Example 4: 2.1.7

The table shows the position of a motorcyclist after accelerating from rest.

t (seconds)	0	1	2	3	4	5	6
s (feet)	0	4.9	20.6	46.5	79.2	124.8	176.7

a. Find the average velocity for each time period:

  i. $[2, 4]$

  ii. $[3, 4]$

  iii. $[4, 5]$

  iv. $[4, 6]$

b. Use the graph of s as a function of t to estimate the instantaneous velocity when $t = 3$.

Solution