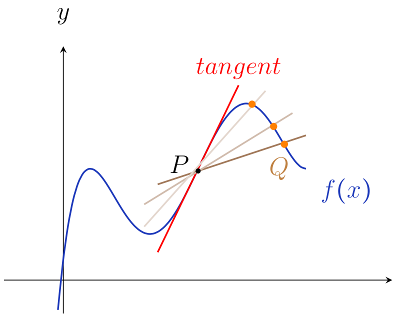
2.2: The Limit of A Function

In the previous lecture, we saw that the slope of the tangent line at point P is the limit of the slopes of the secant line, $\overrightarrow{PQ}$, as Q approaches P.

$$\lim_{Q \to P} m_{PQ} = m_{tangent}$$



This idea can be extended to determine what happens to $f(x)$ as x approaches a (i.e. the limit of a function).

The Limit of A Function

Definition 1: The Limit of A Function

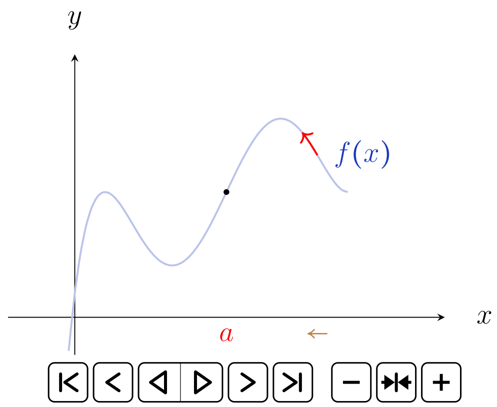
Suppose that $f(x)$ is defined when x is near the number a. This means that f is defined on some open interval that contains a, except possibly at a itself.

Then we write

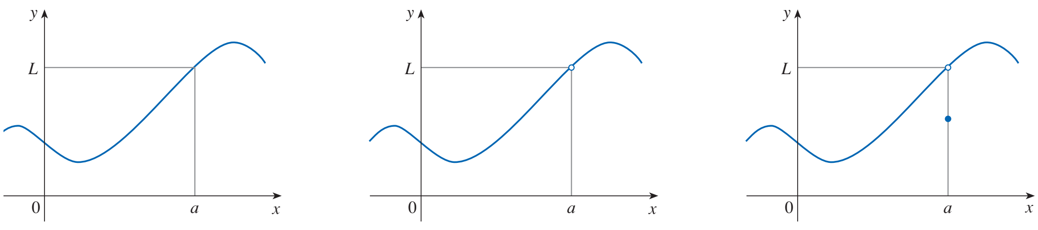
$$\lim_{x \to a} f(x) = L$$

and say *'the limit of $f(x)$ as x approaches a, equals L'*

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.



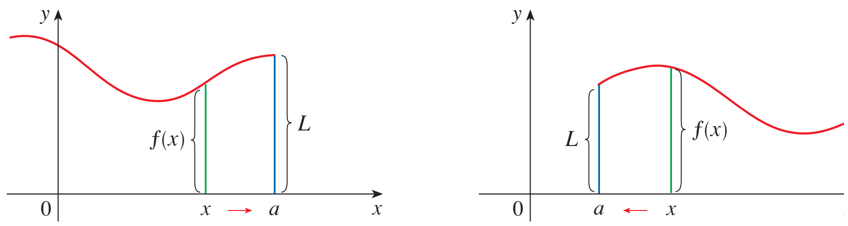
Note that the limit of the function can exist, even though the function may be undefined at $x = a$.



$$\lim_{x \to a} f(x) = L \qquad \lim_{x \to a} f(x) = L \qquad \lim_{x \to a} f(x) = L$$
$$f(a) = L \qquad f(a) = und \qquad f(a) = m$$

One-sided Limits

A one-sided limit only considers values of a function that approaches it from either above or below.



$$\lim_{x \to a^-} f(x) = L \qquad \lim_{x \to a^+} f(x) = L$$

Definition 2: One-sided Limit

The **left-side limit** of a function f, read *'limit of $f(x)$ as x approaches a from the left'* is

$$\lim_{x \to a^-} f(x) = L$$

The **right-side limit** of a function f read *'limit of $f(x)$ as x approaches a from the right'* is

$$\lim_{x \to a^+} f(x) = L$$

The notation $x \to a^-$ indicates that we are only considering values of x that are less than a, and $x \to a^+$ indicates that we are only considering values of x that are greater than a.

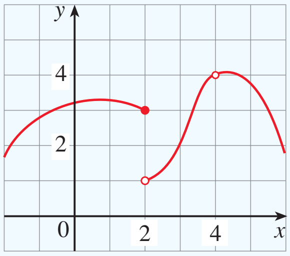
Theorem 1: The General Limit

The general limit of a function, $\lim_{x \to a} f(x) = L$, exists iff the left and right sided limits agree.

$$\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$$

Example 1: 2.2.4

Use the graph of f to state the value of each quantity if it exists. If it does not exist, explain why.



a. $\lim_{x \to 2^-} f(x)$

b. $\lim_{x \to 2^+} f(x)$

c. $\lim_{x \to 2} f(x)$

d. $f(2)$

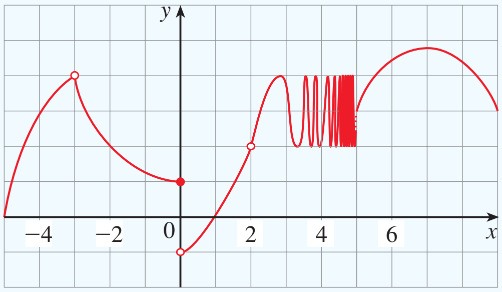
e. $\lim_{x \to 4} f(x)$

f. $f(4)$

Solution

Example 2: 2.2.6

Use the graph of f to state the value of each quantity if it exists. If it does not exist, explain why.



a. $\lim_{x \to -3^-} h(x)$

b. $\lim_{x \to -3^+} h(x)$

c. $\lim_{x \to -3} h(x)$

d. $h(-3)$

e. $\lim_{x \to 0^-} h(x)$

f. $\lim_{x \to 0^+} h(x)$

g. $\lim_{x \to 0} h(x)$

h. $h(0)$

i. $\lim_{x \to 2} h(x)$

j. $h(2)$

k. $\lim_{x \to 5^+} h(x)$

l. $\lim_{x \to 5^-} h(x)$

Solution

Example 3: 2.6.20

Estimate the value of the limit (if it exists) by evaluating the function at the given numbers, correct to four decimal places.

$$\lim_{x \to -1} \frac{x^2 - 2x}{x^2 - x - 2}$$

$x = 0, -0.5, -0.9, -0.99, -0.999, -0.9999,$
$-2, -1.5, -1.1, -1.01, -1.001, 1.0001$

Solution

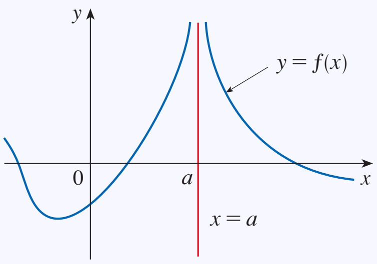
Infinite Limits

Definition 3: Infinite Limits

Suppose that $f(x)$ is defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

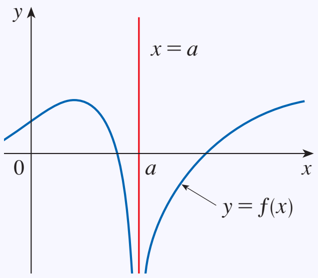
means that the values of $f(x)$ can be made as arbitrarily large by choosing x sufficiently close to a but not equal to a.



Similarly,

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made as arbitrarily large negative by choosing x sufficiently close to a but not equal to a.



Definition 4: Vertical Asymptotes

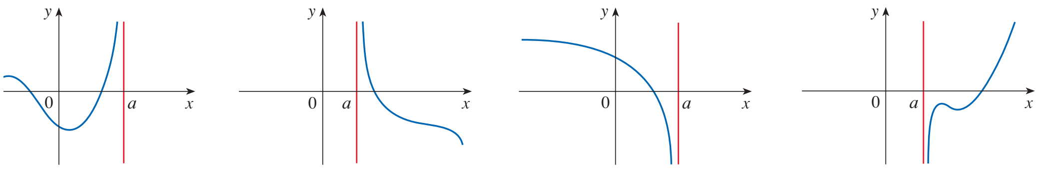
The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ is at least one of the following statements is true

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

$$\lim_{x \to a} f(x) = +\infty \qquad \lim_{x \to a^-} f(x) = +\infty \qquad \lim_{x \to a^+} f(x) = +\infty$$

means that the values of $f(x)$ can be made as arbitrarily large negative by choosing x sufficiently close to a but not equal to a.

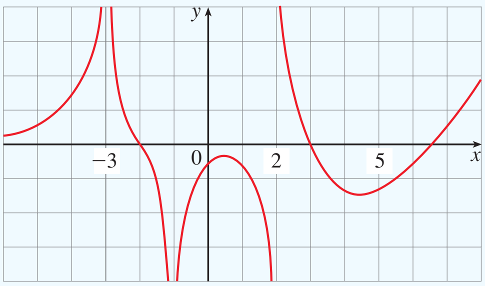
The graphs below illustrate how functions behave near vertical asymptotes.



$\lim_{x\to a^-} f(x) = +\infty$ $\lim_{x\to a^+} f(x) = +\infty$ $\lim_{x\to a^-} f(x) = -\infty$ $\lim_{x\to a^+} f(x) = -\infty$

Example 4: 2.2.8

For the function $R(x)$ whose graph is shown below, state the following



a. $\lim_{x\to -3} R(x)$

b. $\lim_{x\to -1} R(x)$

c. $\lim_{x\to 2^-} R(x)$

d. $\lim_{x\to 2^+} R(x)$

e. $\lim_{x\to 2} R(x)$

f. The equations of the vertical asymptotes.

Solution

Example 5: 2.2.12

Sketch the graph of the function and use it to determine the values of a for which $\lim_{x \to a} f(x)$ exists.

$$f(x) = \begin{cases} \sqrt[3]{x} & ; \quad x \leqslant -1 \\ x & \text{if} \ -1 < x \leqslant 2 \\ (x-1)^2 & ; \quad x > 2 \end{cases}$$

Solution

Example 6: 2.2.24

Use a table of values to estimate the value of the limit.

$$\lim_{p \to -1} \frac{1 + p^9}{1 + p^{15}}$$

Solution

Example 7: 2.2.26

Use a table of values to estimate the value of the limit.

$$\lim_{x \to 0^+} x^2 \ln x$$

Solution

Example 8: 2.2.28

Determine the infinite limit

$$\lim_{x \to 5^-} \frac{x+1}{x-5}$$

Solution

Example 9: 2.2.32

Determine the infinite limit

$$\lim_{x\to 3^-} \frac{\sqrt[3]{x}}{(x-3)^5}$$

Solution

Example 10: 2.2.34

Determine the infinite limit

$$\lim_{x\to 0^+} \ln(\sin x)$$

Solution

Example 11: 2.2.36

Determine the infinite limit

$$\lim_{x \to \pi^-} \cot x$$

Solution

Example 12: 2.2.38

Determine the infinite limit

$$\lim_{x \to 3^+} \frac{x^2 + 4x}{x^2 - 2x - 3}$$

Solution

Example 13: 2.2.40

Determine the infinite limit

$$\lim_{x \to 0^+} \left(\frac{1}{x} - \ln x \right)$$

Solution

Example 14

Find the vertical asymptote(s) of the function

$$y = \frac{x^2 + 1}{3x - 2x^2}$$

Solution