Extra Practice L12 - L15: Solutions

1. **Succulents**

Global demand for brightly coloured succulents is now so high that crime rings are illegally trafficking rare varieties.

Suppose the weight of these succulents is normally distributed with a mean of 220 grams and a standard deviation of 15 grams.

(a) What is the probability that a randomly selected succulent weighs less than 225 grams?

Solution

Let X = the weight of a succulent
X is normally distributed with $\mu = 220$ and $\sigma = 15$

$$P(X < 225) = P\left(Z < \frac{225 - 220}{15}\right) = P(Z < 0.33) = 0.6293$$

(b) If a random sample of 60 succulents were selected. What is the probability that the sample mean weight will be greater than 223 grams?

Solution

Let $\overline{X}$ = the average weight of the succulents;
$\overline{X}$ is normally distributed with $\mu_{\overline{x}} = \mu = 220$ and $\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{n}} = \frac{15}{\sqrt{60}}$

$$P(\overline{X} > 223) = P\left(Z > \frac{223 - 220}{15/\sqrt{60}}\right) = P(Z > 1.55) = 0.0606$$

(c) If a random sample of 70 succulents were selected. What is the probability that the sample mean weight will be between 217 and 223 grams?

Solution

Let $\overline{X}$ = the average weight of the succulents;
$\overline{X}$ is normally distributed with $\mu_{\overline{x}} = \mu = 220$ and $\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{n}} = \frac{15}{\sqrt{70}}$

$$\begin{aligned} P(217 < \overline{X} < 223) &= P\left(\frac{217-220}{15/\sqrt{70}} Z < \frac{223-220}{15/\sqrt{70}}\right) \\ &= P(-1.67 < Z < 1.67) \\ &= 0.9525 - 0.0475 \\ &= 0.9050 \end{aligned}$$

2. **Capuchin Monkeys**

Capuchin monkeys test their social bonds by poking each other in the eye.

A primatologist studying capuchin monkeys collects data on the weight (in kilograms) of adult capuchins in a particular forest. It is known that the weights are normally distributed with a mean of $\mu = 3.5$ kilograms and a standard deviation of $\sigma = 0.6$ kilograms.

(a) What is the probability that a randomly selected capuchin monkey weighs less than 4.2 kilograms?

Solution

Let X = the weight of a capuchin monkey;
X is normally distributed with $\mu = 3.5$ and $\sigma = 0.6$

$$P(X < 4.2) = P\left(Z < \frac{4.2-3.5}{0.6}\right) = P(Z < 1.17) = 0.8790$$

(b) Suppose that there are 10 capuchin monkeys at the zoo. What is the probability that fewer than three of them weigh less than 4.2 kilograms?

Solution

Let Y = number of monkeys weighing less than 4.2
$Y \sim \text{Binomial}(n = 10, p = 0.8790)$

$$\begin{aligned} P(Y < 3) &= P(Y = 0) + P(Y = 1) + P(Y = 2) \\ &= C_0^{10}(0.121)^{10} + C_1^{10}(0.879)(0.121)^9 + C_1^{10}(0.879)^2(0.121)^8 \\ &= 6.7275 \times 10^{-10} + 4.8872 \times 10^{-8} + 1.5976 \times 10^{-6} \\ &= 1.6472 \times 10^{-6} \end{aligned}$$

(c) A random sample of 20 capuchin monkeys are selected. What is the probability that the sample mean weight will be greater than 3.55 kilograms?

Solution

Let $\overline{X}$ = the average weight of the 20 monkeys;
$\overline{X}$ is normally distributed with $\mu_{\overline{x}} = \mu = 3.5$ and $\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{n}} = \frac{0.6}{\sqrt{20}}$

$$P(\overline{X} > 3.55) = P\left(Z > \frac{3.55 - 3.5}{0.6/\sqrt{20}}\right) = P(Z > 0.373) = 0.3544$$

(d) A random sample of 20 capuchin monkeys are selected. What is the probability that the sample mean weight will be between 3.33 and 3.66 kilograms?

Solution

Let $\overline{X}$ = the average weight of the 20 monkeys;
$\overline{X}$ is normally distributed with $\mu_{\overline{x}} = \mu = 3.5$ and $\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{n}} = \frac{0.6}{\sqrt{20}}$

$$\begin{aligned} P(3.33 < \overline{X} < 3.66) &= P\left(\frac{3.33 - 3.5}{0.6/\sqrt{20}} < Z < \frac{3.66 - 3.5}{0.6/\sqrt{20}}\right) \\ &= P(-1.26 < Z < 1.26) \\ &= 0.8962 - 0.1038 \\ &= 0.7924 \end{aligned}$$

3. **Vandals**

In 1980s San Francisco, gangs of vandals were hired to slash train seats to ensure repair teams got more work. The cushions were cut in specific patterns so the repairs company knew who to pay.

A transit investigator collects a random sample of 40 train seats and measures the depth of the cuts (in millimeters). The sample yields an average cut depth of 12.8 mm. From previous investigations, the population standard deviation is known to be 3.5 mm.

(a) Construct and interpret a 95% confidence interval for the true mean depth of cuts made by these hired vandals.

Solution

Let $\overline{X}$ = the average cut depth from a sample of $n = 40$ seats.
$\because \sigma$ is known $\quad \Rightarrow \quad Z_{0.05} = \pm 1.96$

$$\begin{aligned} \overline{X} &\pm Z_{0.05} \cdot \frac{\sigma}{\sqrt{n}} \\ 12.8 &\pm 1.96 \cdot \frac{3.5}{\sqrt{40}} \\ 12.8 &\pm 1.0847 \\ 11.7153 < &\mu < 12.8847 \end{aligned}$$

Conclusion: With repeated sampling, we are 95% confident that the actual average depth of the cuts made is between 11.7153mm and 12.8847mm.

(b) Suppose a repair company claims that the average depth of the seat cuts made by hired vandals was at least 12 millimeters. Based on the 95% confidence interval you constructed, can this claim be validated? Justify your answer.

Solution

Yes, since $12 \in CI$ we can conclude that the average depth of the cuts made is at least 12mm.

(c) If the investigator wants the margin of error for the 95% confidence interval to be no more than 0.5 mm, what is the minimum sample size needed?

Solution

$$n = \left(\frac{Z_c \cdot \sigma}{E}\right)^2 = \left(\frac{1.96 \cdot 3.5}{0.5}\right)^2 = 188.2384$$

$\Rightarrow$ 189 seats are required.

4. **Octopuses**

Octopuses can hold years-long grudges against keepers they don't like, lying in wait to ambush them with jets of water whenever they see them.

At a marine research center, a biologist records the number of ambushes by each of 12 octopuses against their least favorite keeper over a one-month period. The data yields a sample mean of 8.2 ambushes, with a sample standard deviation of 2.5 ambushes.

(a) Construct and interpret a 90% confidence interval for the true mean number of ambushes held in grudge by octopuses.

Solution

Let $\overline{X}$ = the average number of ambushes;
σ is **unknown** $\Rightarrow$ use the t-distribution.

$$\begin{aligned}\overline{X} &\pm t_{0.05,11}\cdot\frac{s}{\sqrt{n}}\\ 8.2 &\pm 1.796\cdot\frac{2.5}{\sqrt{12}}\\ 8.2 &\pm 1.296\\ 6.904 < \mu &< 9.496\end{aligned}$$

Interpretation: We are 90% confident that the true mean number of ambushes an octopus directs at a disliked keeper in one month is between 6.904 and 9.496.

(b) Construct and interpret a 95% confidence interval for the true mean number of ambushes held in grudge by octopuses.

Solution

$$\begin{aligned}\overline{X} &\pm t_{0.025,11}\cdot\frac{s}{\sqrt{n}}\\ 8.2 &\pm 2.201\cdot\frac{2.5}{\sqrt{12}}\\ 8.2 &\pm 1.588\\ 6.612 < \mu &< 9.788\end{aligned}$$

Interpretation: We are 95% confident that the true mean number of ambushes an octopus directs at a disliked keeper in one month is between 6.612 and 9.788.

(c) Construct and interpret a 99% confidence interval for the true mean number of ambushes held in grudge by octopuses.

Solution

$$\begin{aligned}\overline{X} &\pm t_{0.005,11}\cdot\frac{s}{\sqrt{n}}\\ 8.2 &\pm 3.106\cdot\frac{2.5}{\sqrt{12}}\\ 8.2 &\pm 2.241\\ 5.959 < \mu &< 10.441\end{aligned}$$

Interpretation: We are 99% confident that the true mean number of ambushes an octopus directs at a disliked keeper in one month is between 5.959 and 10.441.

(d) Consider the three intervals above. Which interval is the most precise? Why?

Solution

The most precise of these three intervals is the 90% confidence interval since it has the smallest margin of error.

5. **Forks**

Forks were banned for sailors in the British Navy as late as the 1890s because they were considered 'unmanly'.

A historian studying old naval records randomly selects $n = 180$ sailors from ships in the 1880s and finds that $x = 115$ of them reported using only knives and spoons during meals.

(a) Construct and interpret a 98% confidence interval for the true proportion of British Navy sailors in the 1880s who did not use forks.

Solution

Let $X =$ the number of sailors who did not use forks.

$\hat{p} = \frac{x}{n} = \frac{115}{180} \approx 0.6389$ and $Z = 2.33$ for a 98% confidence level.

$$\begin{aligned}
\hat{p} \pm Z \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \\
0.6389 \pm 2.33 \cdot \sqrt{\frac{0.6389(1-0.6389)}{180}} \\
0.6389 \pm 0.0831 \\
0.5558 < p < 0.7220
\end{aligned}$$

Interpretation: We are 98% confident that the true proportion of British Navy sailors in the 1880s who did not use forks is between 55.58% and 72.20%.

(b) A historian wants to estimate the true proportion of sailors who did not use forks with a 95% confidence level and a margin of error of no more than 4%. A preliminary study suggests that about 65% of sailors avoided using forks. What is the minimum sample size the historian should use?

Solution

Let $p = 0.65$, $E = 0.04$, and for a 95% confidence level, $Z = 1.96$

$$n = \hat{p} \cdot (1 - \hat{p}) \cdot \left(\frac{Z}{E}\right)^2 = 0.65 \cdot (0.35) \cdot \left(\frac{1.96}{0.04}\right)^2 = (23.393)^2 = 547.2$$

The historian should sample at least 548 sailors.

(c) Construct and interpret a 98% lower confidence bound for the true proportion of British Navy sailors in the 1880s who did not use forks.

Solution

We use the same values as in part (a), with a one-sided $Z = 2.05$ for a 98% lower bound.

$$\begin{aligned} \hat{p} - Z \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} &\leq p \\ 0.6389 - 2.05 \cdot \sqrt{\frac{0.6389(1-0.6389)}{180}} &\leq p \\ 0.5655 &\leq p \end{aligned}$$

Interpretation: We are 98% confident that at least 56.55% of British Navy sailors in the 1880s did not use forks.