L4. Probability Theory

Sample Spaces and Events

Example 1: DeepMind

DeepMind, Google's AI powerhouse, has cracked protein folding and advanced cancer detection—but UK high school math? Not so much. After months of training on millions of examples, it barely scraped by with an E.

For context, British grades run from A*, A, B,C,D, and E. To get into a decent university you need a mix of A's and B's.

Let $S = \{A^*, A, B, C, D, E\}$
T_i = the grades obtained by the AI on three separate runs during the i^{th} training day.

Consider the following sets:

$$T_1 = \{B, C, E\} \qquad T_2 = \{C, D, E\} \qquad T_3 = \{A, B, E\}$$

Compute the following:

a. $T_1 \cap T_2$

b. $T_1 \cup T_2$

c. T_1'

d. $(T_1 \cup T_3)'$

e. $T_1' \cap T_3'$

f. $(T_2 \cup T_3)' \cap T_1$

Solution

Example 2: GPS Spoofing

GPS spoofing tricks receivers into showing fake locations, and Russia is basically the world champion at it — using it on both enemies *and* its own people. The problem went unnoticed until *Pokémon Go* players near the Kremlin found themselves mysteriously teleported miles away from their desired location.

After enough confused Pikachu hunters complained, Nintendo patched the game to ignore the Kremlin's shenanigans—finally, a win against Russian interference.

Let $S = \mathbb{R}^+$ be the sample space representing the level of distortion (in km) that GPS spoofing creates.

Let $A = [1, 3), B = [3, 5]$, and $C = (2, 4)$ represent the distortion experienced by three players of the game.

Determine

a. $(A \cup B) \cap C$

b. $(A \cup C) \cap B$

c. $(A \cup B) \cap (A \cup C)$

Solution

Example 3: Blame it on The Ambien

After Rosanne Barr posted a series of racist tweets on Twitter, ABC cancelled her sitcom "The Conners". The actress blamed her actions on the sleep medication, Ambien. But Sanofi, the makers of the sedative were quick to point out that although their product may have numerous side effects; racism wasn't one of them.

Let $S = \{t : \text{ time in minutes } | 0 < t\}$
P_i = the time it takes for the i^{th} patient on Ambien to fall asleep after taking the medication.

Consider the following events:

$$P_1 = (15, 20) \qquad P_2 = [15, 45] \qquad P_3 = (30, 60)$$
$$P_4 = [30, \infty) \qquad P_5 = (0, 60]$$

Compute the following:

a. $P_1 \cap P_3$

b. P_3'

c. $P_2 \cap P_4$

d. $P_2 \cup P_4$

e. $(P_4 \cap P_5)' \cap P_2'$

Solution

Example 4: Breaking Bad: Coin Toss of Doom

In Season 1, Episode 2, Walter White and Jesse Pinkman find themselves in a sticky situation - there's a body in a bathtub, and it needs disappearing. They decide to let fate (or, more accurately, a coin toss) determine who has to do the dirty work.

a. List the sample space for one coin toss.

b. List the sample space for three coin tosses.

c. Suppose that Jesse calls "Heads" before every coin toss. Define the following events:

- E_1 : Jesse loses every toss and has to dissolve the body (again).
- E_2 : Walter loses at least once but not every time.
- E_3 : The sequence contains exactly one heads.
- E_4 : The sequence has at least two tails (meaning Jesse is probably doomed).
- E_5 : The sequence is all heads or all tails (pure fate decides everything).

Determine the following:

i $E_1 \cup E_2$ (The set of outcomes where either Jesse loses every time OR Walter loses at least once.)

ii $E_1 \cap E_2$ (The set of outcomes where Jesse loses every time AND Walter loses at least once does this even exist?)

iii $E_1' \cup E_2'$ (The complement of Jesse's full loss combined with the complement of Walter losing at least once.)

iv $E_3 \cup E_5$ (The set of outcomes where there's exactly one heads OR fate rules completely.)

v $E_4 \cap E_2$ (The set of outcomes where Jesse probably loses AND Walter loses at least once.)

Solution

Calculating Probabilities

Example 5: Coin Toss

Many episodes of The Simpsons feature a coin toss - including one where Homer lets a coin toss determine his fate... and keeps flipping until he gets the answer he wants.

Suppose that flip the coin three times. What is the probability that, in the sequence of three coin tosses that

a. exactly one head will occur?

b. at most two heads occur?

c. no heads will appear?

d. no tails will occur?

e. no head or tails occur?

Solution

Example 6: King Henry VIII

King Henry VIII was obsessed with having a male heir—so much so that he went through six wives, all because medieval genetics refused to cooperate. His first wife, Catherine of Aragon, gave birth to several children, but only one—Mary—survived, leading Henry to ditch Catherine and change the course of English history.

Suppose that he and his wife (for now) plan to have four children, praying for at least one boy — before Henry gets impatient and starts looking for a new wife.

a. List the sample space, S, for all possible sequences of boys (B) and girls (G).

b. Find the probability that Henry will have:

   i. A 2 boys, 2 girls split (mild disappointment but acceptable).

   ii. A 1 boy, 3 girls OR 3 boys, 1 girl split (panic mode).

   iii. 4 boys or 4 girls (total triumph or immediate annulment).

Solution

Example 7: Monopoly

Need help de-escalating a Monopoly meltdown this Christmas? There's a hotline for that. In the UK, frustrated players can call 800-689-4903 to speak with a Monopoly expert who settles game-related disputes. Hasbro launched the service after discovering that 51% of Monopoly games end in chaos.

In the game of Monopoly, players roll two dice to determine how many paces they are to move around the board. What is the probability that the

a. sum is twelve?

b. sum is seven?

c. sum is nine or ten?

d. both die show the same number?

e. both die show different numbers?

f. the difference between the two dice is more than two?

Solution