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14.1 R,L,C IN SERIES IN AN AC CIRCUIT.

The scheme in figure 1 shows a resistor R, a capacitor C and an inductor L ***in series*** connected to the terminals of an ***AC source***. As the current is the same for all elements in series, one starts by writing

$$i_R = i_C = i_L = i = i_0 \sin \omega t \qquad (1)$$

As , in general, the voltage and current have a phase shift in AC circuits $\upsilon = \upsilon_0 \sin(\omega t + \phi)$ (2)

The values of υ_0 and ω are fixed by the source while the ***amplitude of current*** "i_o" and the ***phase shift*** ***"φ"*** between the *voltage* and the *current* depend on the numerical values of R, L, C in a circuit. Let's refer to an moment "*t* " when the " *current* ***phasor*** " has the direction shown in figure 2 and let's label as υ, υ_R, υ_C and υ_L the voltage at the source, resistor, capacitor and inductor at same moment. Next, by applying the second rule of Kirchhoff along the ***current direction,*** one get :

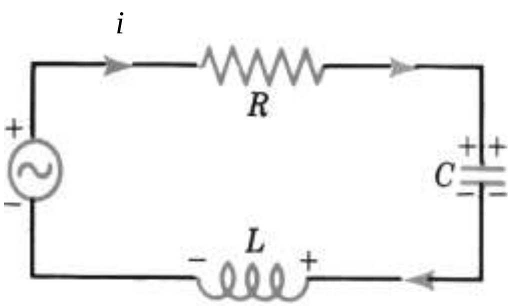

Figure 1

$$\upsilon - \upsilon_R - \upsilon_C - \upsilon_L = 0 \Rightarrow\Rightarrow\Rightarrow \upsilon = \upsilon_R + \upsilon_C + \upsilon_L \qquad (3)$$

As the phase shift between the current and the voltage is different for each element, in general, there is a phase shift between terms in expression (3) and one must consider it as a phasor relationship.

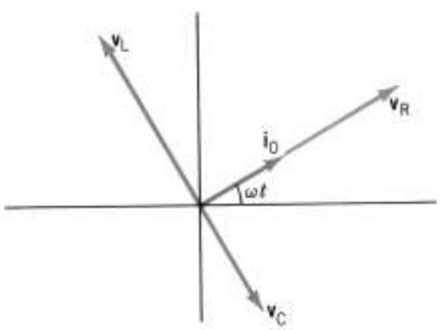

Figure 2

- So, one draws the phasors diagram shown in fig.2 as follows:

1. Start by drawing the current phasor at the moment "t" ($\varphi_i = \omega t$).
2. The phasor for **R-voltage** has zero phase shift versus current.
So, this phasor is drawn along the same direction as *i-phasor*.
3. The phasor for **C- voltage lags** by **π/2** vs. *i-phasor*.
4. The phasor for **L-voltage** is **ahead** by **π/2** vs. *i-phasor*.
Then, the *phasor for applied voltage* by source will be the sum of three *potential drops' phasors* (see fig.3)

$$\upsilon_S^{ph} = \upsilon_R^{ph} + \upsilon_C^{ph} + \upsilon_L^{ph} \qquad (4)$$

Note that the expression (4) is a "*vector sum of vectors"* each with a magnitude equal to the maximum value of corresponding voltage ($\upsilon_{0R}, \upsilon_{0C}, \upsilon_{0L}$). As the C and L phasors align in opposite direction, their "*vector"* sum will be a *phasor with magnitude*[1] $|v_{0L} - v_{0C}|$. So, the sum of three phasors is reduced into the sum of two phasors (figure 3). Now one can express the *amplitude of applied voltage* in circuit by the sum of two other phasors' amplitudes as

$$\upsilon_0^2 = \upsilon_{0R}^2 + (\upsilon_{0L} - \upsilon_{0C})^2 \qquad (5)$$

Using the relations between the ***maxima values*** of *instantaneous voltage and instantaneous current* through R, C and L, relation 5 transforms to

$$\upsilon_0^2 = (i_0 Z)^2 = i_0^2 [R^2 + (X_L - X_C)^2] \qquad (6)$$

The parameter

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \qquad (7)$$

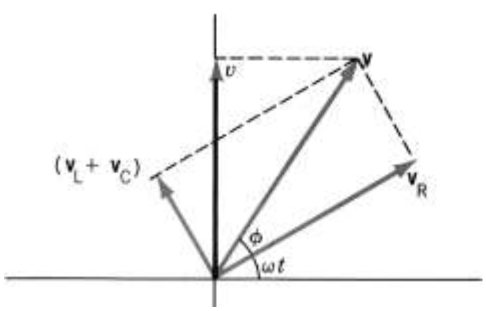

Figure 3

is called the ***impedance*** of R, C, L in **series circuit**. This parameter

[1] To avoid ambiguities, we have assumed that $\upsilon_{0L} > \upsilon_{0C}$.

determines the amplitude of current in an AC circuit; its unit is ohm. Then, the relation (6) can be transformed to Ohm's law relating ***the amplitudes*** of ***applied voltage*** and the ***current*** in the circuit

$$\upsilon_0 = i_0 * Z \qquad (8)$$

By multiplying by 0.707 both sides, one gets the Ohm's law for ***rms*** values $V = I * Z$ (9)

- Knowing υ_0 of AC source , one may use (8) to calculate the ***amplitude of current*** " i_0 " in circuit. Also, from phasor diagram in fig.3, one can find the *phase shift* between *source voltage* and *current* in circuit

$$tan\emptyset = \frac{\left|v_L^{ph} + v_C^{ph}\right|}{\left|v_R^{ph}\right|} = \frac{v_{0L} - v_{0C}}{v_{0R}} = \frac{i_0(X_L - X_C)}{i_0 R} = \frac{X_L - X_C}{R}$$

So, $$\tan\phi = \frac{X_L - X_C}{R} \qquad (10)$$

If $\mathbf{X_L = X_C}$ the phase shift ϕ **= 0;** the current in circuit and the driving potential are all time in phase.
If $\mathbf{X_L > X_C}$ the phase shift $\phi > \mathbf{0}$ which means that the driving ***potential is ahead of current***.
If $\mathbf{X_L < X_C}$ the phase shift $\phi < \mathbf{0}$ which means that the ***current is ahead of driving potential***.

14.2 RESONANCE IN A CIRCUIT WITH RLC *IN SERIES*.

- The ***impedance "Z "*** defines the characteristics of current passing through the circuit when an AC voltage applies on it. One studies the "*average behaviour* " of circuit through ***rms*** values for current, voltage and power. If the ***rms voltage*** of source is ***V***, the ***rms current I*** in a R,L,C in series circuit is

$$I = \frac{V}{Z} = \frac{V}{\sqrt{R^2 + (X_L - X_C)^2}} \qquad (11)$$

The *rms* ***I*** value ***depends on the frequency*** of applied AC voltage because X_L and X_C depend on ω.
As seen from (11), there is a maximum ***rms current*** in circuit if $X_L - X_C = 0$; $I_{\max} = \frac{V}{R}$ (12)

As $X_L = X_C \rightarrow \omega_0 L = \frac{1}{\omega_0 C} \Rightarrow \omega_0^2 = \frac{1}{LC} \Rightarrow\Rightarrow \omega_0 = \frac{1}{\sqrt{LC}}$, the ***frequency of applied voltage*** which produces I_{max} ***(i.e. a resonance)*** in circuit is $$\omega = \omega_0 = \frac{1}{\sqrt{LC}} \qquad (13)$$

Remember that $\omega_0 = \frac{1}{\sqrt{LC}}$ is the *circular frequency* of a *C, L circuit without damping*(R = 0).

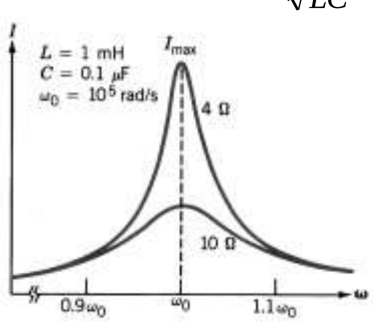

Figure 4

The graphs in fig.4 show the evolution of ***rms current*** in circuit when the *frequency of applied AC voltage is close to the resonance frequency of circuit* ω_0. Note that the relation (11) shows that ***at resonance***, an AC circuit behaves as if containing only a resistor. At ***resonance*** φ = 0 and the ***voltage*** and ***current*** are ***in phase*** which means that the *phasors* of source voltage and current are aligned following the same direction at any moment of time.
As shown in fig.4, for the same C, L values, the ***resonance curve*** of a L, C, R circuit becomes ***sharper*** (fig.4) for ***smaller values of R***.

14.3 POWER IN AC CIRCUITS

- The instantaneous power delivered by an AC source in a circuit is a function of time

$$p = i * \upsilon = i_0 \sin(\omega t) * \upsilon_0 \sin(\omega t + \phi) = (i_0 \upsilon_0) \sin(\omega t)[\sin(\omega t)\cos\phi + \cos(\omega t)\sin\phi)] =$$

$$p = i_0 \upsilon_0 [\sin^2(\omega t)\cos\phi + \sin(\omega t)\cos(\omega t)\sin\phi] = i_0 \upsilon_0 \{\sin^2(\omega t)\cos\phi + \frac{[2\sin(\omega t)\cos(\omega t)]}{2}\sin\phi]$$

$$p = i_0 \upsilon_0 [\sin^2(\omega t)\cos\phi + 0.5 * \sin(2\omega t)\sin\phi] \qquad (14)$$

For practical considerations, one refers to average value of power and this average is referred to one period of oscillation. The ***average power delivered by the AC source in circuit during one period*** is

$$p_{av} = \frac{1}{T}\int_0^T p dt = i_0 \upsilon_0 \cos\phi \frac{1}{T}\int_0^T \sin^2(\omega t) dt + i_0 \upsilon_0 0.5 \sin\phi \frac{1}{T}\int_0^T \sin(2\omega t) dt \qquad (15)$$

$$\int_0^T sin^2(\omega t)dt = \left(\frac{t}{2} - \frac{\sin(2\omega t)}{4\omega}\right)_0^T = \left(\frac{T}{2} - \frac{\sin(2\omega T)}{4\omega}\right) - 0 = \left(\frac{T}{2} - \frac{\sin(4\pi)}{4\omega}\right) = \frac{T}{2}$$

$$\int_0^T \sin(2\omega t)\, dt = -\frac{1}{\omega}(cos\omega t)\ |_0^T = -\frac{1}{\omega}\left(\cos\frac{2\pi}{T}T - cos0\right) = -\frac{1}{\omega}(cos2\pi - cos0) = -\frac{1}{\omega}(1-1) = 0$$

As the first integral is equal to "***T/2*** "and the second integral is equal ***zero***, it comes out that the average *power delivered by an AC source in circuit* is

$$p_{av} = i_0 v_0 cos\phi * \frac{1}{T} * \frac{T}{2} = \frac{1}{2} i_0 v_0 cos\phi \qquad (16)$$

This expression shows that the phase shift between the applied voltage and the current in circuit *affects* directly the *average power delivered* by the source in a circuit.

-If the circuit contains only a set of R, C, L <u>*in series*</u>, one may use the phasor diagram in figure 3 to get the cosine of phase shift between current and source voltage as follows

$$\cos\phi = \frac{\left|\upsilon_R^{ph}\right|}{\left|\upsilon_S^{ph}\right|} = \frac{\upsilon_{0R}}{\upsilon_0} \qquad (17)$$

and $$v_0 cos\phi = v_0 * \frac{v_{0R}}{v_0} = v_{0R} = i_0 R \qquad (18)$$

Then by substituting this in expression (16) $$p_{av} = \frac{1}{2} i_0 v_{0R} = \frac{1}{2} i_0^2 R = \frac{i_0}{\sqrt{2}} * \frac{i_0}{\sqrt{2}} R = I^2 R \qquad (19)$$

Expression (19) shows that the average power supplied by AC source in a R,L,C <u>*circuit in series*</u> is the same as that dissipated as heat in resistor R by a current equal to *I rms*, i.e. $p_{av} = P_R = I^2 R$

- In general, disregarding the elements in a circuit, one may rewrite the expression (16) in the form

$$p_{av} = \frac{i_0}{\sqrt{2}}\frac{\upsilon_0}{\sqrt{2}}\cos\phi = IV\cos\phi = P \qquad (20)$$

At this expression the *average power supplied by the source is expressed through **rms** values of* ***<u>source voltage</u>*** *and* ***current*** in circuit**.** So, simply by measuring the ***rms*** values of ***current*** and **voltage** at ***source terminals and their phase shift*** one may calculate the ***average power*** delivered by an AC source, even ***without knowing the elements of the circuit*** where is supplied this power.

- The value of " ***cosφ*** " factor in expression $P = IV\cos\phi$ depends only on the phase shift between the current in circuit and source voltage. It affects very much (*can change it from zero to maximum value)* the power released by source in a circuit. For this reason one has named " ***cosφ*** " as ***power factor***.

If ***cosφ = 0*** , ***φ =± π/2*** which means that *the* ***load is purely inductive or capacitive.*** In this situation, the energy delivered in circuit during a half-period is returned into the source during the other half of period when the current inverts the flowing direction. Then, *the average power* provided by the source in circuit during a period is ***zero***$(p_{av}=P=0)$.

If ***cosφ = 1*** one gets ***φ =0*** which means that *the* ***load is purely resistive****; all the energy sent into the circuit by the source is dissipated as heat at the resistor. It does not return any more into the source.*

Remember that, for known values of R, L, C *in series*, one may find the " *phase shift* ϕ" from relation

$$\tan\phi = \frac{X_L - X_C}{R} = \frac{\omega L - 1/\omega C}{R} \qquad (21)$$

- Let's consider another time the expression (19) and substitute there $I = \frac{V}{Z} = \frac{V}{\sqrt{R^2 + (X_L - X_C)^2}}$

Then, we can calculate

$$p_{av} = I^2 R = \left(\frac{V}{Z}\right)^2 R = \frac{V^2 R}{R^2 + (X_L - X_C)^2} \qquad (22)$$

Expression (22) tells that the ***average power delivered in circuit R,L,C in series*** depends on the ***source frequency;*** it is a maximum $p_{av}^{max} = V^2/R$ (23) (***there is resonance***) when $\omega = \omega_0 = \frac{1}{\sqrt{LC}}$ (24)

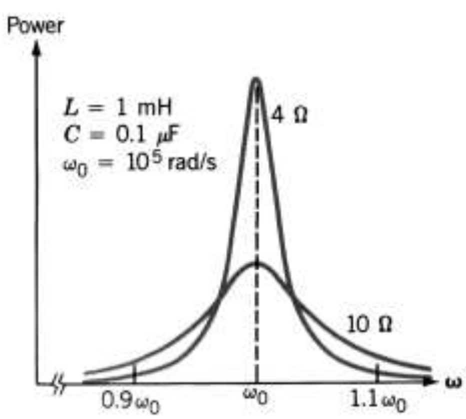

Figure 5

One uses the resonance of delivered power in R , C, L circuits for the reception of radio and TV signals. When a package of electric signals with different frequencies is applied at R, L, C circuit input, only those with frequency inside the resonance region "*are allowed* " to build up a current at their frequency through circuit and this way deliver their power into circuit. One should use small values of R to get sharp resonance curves(fig.5). Such R, C, L in series circuits are able to provide very selective reception by *distinguishing* and *rejecting* even *signals* with frequency *close to* $\boldsymbol{\omega_0}$. This feature is widely used to build sharp ***filters for EM signals***.