LECTURE 12

INTRODUCTION

-Until now, we were referring to the ***model of object as a single material point*** (i.e. a particle with object's mass) and have studied the motion of object by analyzing the motion of this particle in space. This model is valid if **all points of object** have the **same vector of displacement** $\overrightarrow{\Delta s}$. This condition is fulfilled in case of translational motion and even for rotational motion if the distance of object from the axe of rotation is much larger than the object dimensions (as ***the distances of all object points from the rotation axe are practically equal)***.

In this chapter one deals with the rotational motion around an **axis passing through** the **object** or **close** to it. In these circumstances the **displacement vector is different for each point of object.** Therefore one **cannot** model the object motion as **a single material point motion**. In those situations, one has to model the whole object as ***a set of particles*** *that* ***rotate simultaneously together*** *around the same axis of rotation.*

- The ***rigid body*** is an object which **shape** remain **unchanged during the motion**, i.e. the relative distances between particles that constitute the object do not change. Let's consider the ***rotation of a rigid body*** *around* a ***fixed axis*** in an ***inertial frame Oxyz***. In this case, the axis of rotation does not move versus the frame of reference while the body rotates around it (fig.1). *In general, one places Oz axe along the axis of rotation.*

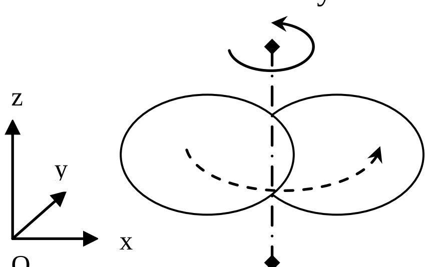


Figure 1

In these circumstances all body **particles move on circular paths centered on the rotation axis;** this is a *pure rotational motion*. In a *general rotational motion*, the *axis of rotation can move with respect to the reference frame* but it remains always *fixed with respect to the body*. One studies the *general rotation* as a *composed motion* constituted by a pure rotation and a translation motion.

1] ROTATIONAL KINEMATICS IN A PURE ROTATIONAL MOTION

-How to describe the motion of a rigid body particles during a *pure rotational motion*? At first, one notes that during such type of rotation, **all particles** of object **rotate** by the **same angle** around **same rotation axis** even though their **displacement vectors are different**. Let's consider the pure CCW rotation of a body around an axe passing through it (by point O and perpendicular to page , fig.2) during a given interval of time.

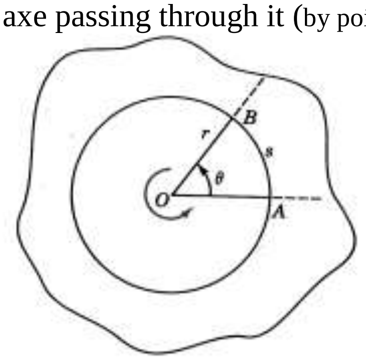


Figure 2
A section of body perpendicular to rotation axis

During this interval of time all particles on the direction OA get rotated by the same angle θ. Meanwhile, the travelled distance for each of them depends on the distance from the axe (point O). For point *A at distance " r "* from the axis of rotation, the length of *travelled distance " s "* is

$$s = r * \theta \qquad (1)$$

(***θ** in **radian,*** counted versus a fixed reference direction "*say OA"**, taken "+" for CCW***).
Assuming that the body rotates by the angle ***Δθ*** during the short interval of time ***Δt,*** one defines its ***average angular velocity*** as

$$\omega_{av} = \frac{\Delta\theta}{\Delta t} = \frac{\theta_f - \theta_i}{t_f - t_i} \qquad (2)$$

If the rotation rate changes one uses the ***instantaneous angular velocity***

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \qquad (3)$$

The *angular velocities* (**ω_{av}, ω**) give the number of radians covered in unit time; its SI unit is ***rad / sec***.

- In fact, the **angular velocity is a vector** that informs about the *sense of rotation*, too. Its **direction** is defined by the **right hand rule;** if one curls the *four fingers of right hand* following the rotation sense of object, the thumb gives the direction of vector $\vec{\omega}$. The vector $\vec{\omega}$ is **placed on the rotation axis**. Like in all physics studies, one should start by defining a reference frame of axes. For ***pure rotations,*** one places ***Oz*** axis ***along the rotation axis*** and ***Ox, Oy perpendicular to Oz.*** Then, if when *looking down the Oz axis*, the object rotates ***counter clockwise*** *"CCW"* one considers the rotation to be ***positive*** (case of fig.3) and the vector $\vec{\omega}$ is aligned along positive direction of Oz. Otherwise, $\vec{\omega}$ is aligned in opposite direction to Oz, the rotation is ***clockwise*** "CW" and it is considered to be ***negative***. The expressions (2, 3) give the component of vector $\vec{\omega}$ on Oz axis.

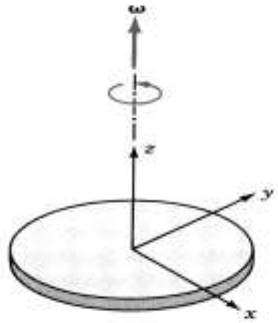


Figure 3

- If the *magnitude of angular velocity* is constant (***ω = c<u>te</u>***) the rotation is ***uniform***. One defines the ***period "T"*** of an ***uniform rotation*** as the *time for one revolution of object.* From the expression (2) one can get $\Delta t = \Delta\theta/\omega_{av} = \Delta\theta / \omega$. As for one revolution $\Delta t = T$ and $\Delta\theta = 2\pi$, it comes out that

$$T = \frac{2\pi}{\omega} \qquad (4)$$

One has defined the frequency of rotations "***f*** " [**Hz** or **sec**$^{-1}$] as the ***number of revolutions*** in ***one second***.

So, for uniform rotations, one gets $f = \frac{1}{T}$ (5) and $\omega[rad/\sec] = \frac{2\pi}{T[\sec]} = 2\pi f$ (6)

-The *instantaneous velocity* of a particle is defined by its displacement "***ds***" for an infinitesimal time "***dt*** ", as $\upsilon = \frac{ds}{dt}$. For a particle at distance " ***r*** " from rotation axis, and an infinitesimal rotation by "***dθ*** " the relation (1) would give its *infinitesimal displacement* as $ds = r * d\theta$ (7)

Then, the *instantaneous velocity* for this particle would be $\upsilon(r) = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega$ (8)

Expression (8) shows that the ***speed of object's particles*** (i.e. magnitude of velocity) increases with the distance "***r***" from the axis of rotation. Also, all particles at same distance "***r***" from rotation axis move at same *speed*.

- For situations where the ***angular velocity*** is not constant (i.e. ***non-uniform rotations***), one has to use the ***angular acceleration***. One has defined ***average angular acceleration*** as $\alpha_{av}[rad/s^2] = \frac{\Delta\omega}{\Delta t}$ (9)

and the ***instantaneous angular acceleration*** as $\alpha[rad/s^2] = \lim_{\Delta t\to 0}\alpha_{av} = \lim_{\Delta t\to 0}\frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$ (10)

Actually, the **angular acceleration** of a rotation **is a vector** $\vec{\alpha}$ **that lays on rotation axis Oz.** So, it has only one component "$\alpha_z = \alpha$". If $\boldsymbol{\alpha_z}$ has the ***same sign*** (+ or -) as $\boldsymbol{\omega_z}$, this means that $\vec{\alpha}$ has the **same sense** as $\vec{\omega}$ and the rotation is ***speeding up***; if they have **opposite sign** the rotation is ***slowing down***.

- For ***<u>constant angular acceleration,</u> "α" is constant*** *and* one can rewrite expression (9) as

$\Delta\omega = \alpha_{av}\Delta t \equiv \alpha\Delta t$. Next, taking $t_i = 0$ _ $\Delta t = t_f - t_i = t_f \equiv t$ one gets $\omega - \omega_0 = \alpha * t$ (11)

This relation has the same mathematical form as that of translational velocity in 1D motion with **constant acceleration**. By using the same area technique as in 1D *kinematics* (see lecture 4), one may find out that

the ***angle of rotation*** is expressed as $$\theta = \theta_0 + \omega * t + \frac{1}{2}\alpha * t^2 \qquad (12)$$

and the ***angular velocity*** is expressed as $$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0) \qquad (13)$$

So, for a motion at ***<u>CONSTANT</u>***

Translational Acceleration (a = c^te^)	***Angular acceleration (α = c^te^)***
$\upsilon = \upsilon_0 + a * t$	$\omega = \omega_0 + \alpha * t$
$x - x_0 = \upsilon_0 * t + \frac{1}{2} a * t^2$	$\theta - \theta_0 = \omega_0 * t + \frac{1}{2}\alpha * t^2$
$x - x_0 = \frac{\upsilon_0 + \upsilon}{2} * t$	$\theta - \theta_0 = \frac{\omega_0 + \omega}{2} * t$
$\upsilon^2 = \upsilon_0^2 + 2a(x - x_0)$	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$

-If object rotates **uniformly** ($\omega = c^{te}$; $\alpha = 0$), any particle at distance "***r***" from rotation axe moves at same ***constant velocity υ(r)*** given by (8). A particle that rotates uniformly on a circular path with radius "***r***" has a centripetal (*or radial*) acceleration $a_{c-r} = \frac{\upsilon_r^2}{r}$, which, by using expression (8), can be written as

$$a_{c-r} = \frac{\upsilon_r^2}{r} = \frac{\omega^2 r^2}{r} = \omega^2 * r \qquad (14)$$

This quantity (*centripetal* ***or radial acceleration***) is a ***vector*** $\vec{a}_{c-r}$ ***directed versus the center of rotation***.

In case of ***accelerated circular motion***, the *angular velocity* ***ω*** *changes* in time (see 11). Then, relation (8) tells that the magnitude of *velocity vector* $\vec{\upsilon}$, will change, too. This $\vec{\upsilon}$ - change is directed *along the tangent* to the circular path and produces a *tangential acceleration* with magnitude (*derivative of 8, see 10*)

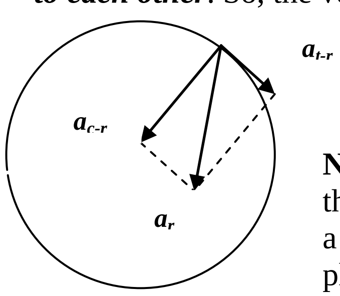
$$a_{t-r} = \frac{d\upsilon}{dt} = \frac{d(\omega r)}{dt} = r\frac{d\omega}{dt} = \alpha * r \qquad (15)$$

In a ***non-uniform*** rotation, a particle at distance "*r* " moves at a ***net*** **(***translational***)** ***acceleration***

$$\vec{a}_r = \vec{a}_{c-r} + \vec{a}_{t-r} \qquad (16)$$

The ***centripetal and tangential accelerations*** lay on the plane of particle rotation and are ***perpendicular to each other***. So, the vector of ***<u>net acceleration</u>*** lies on the same plane (fig 4) and has magnitude

$$a_r = \sqrt{a_{c-r}^2 + a_{t-r}^2} \qquad (17)$$

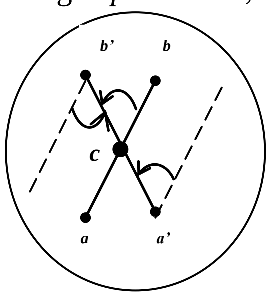


NOTE: One uses the term *angular velocity* and *angular acceleration* to distinguish them from *translational velocity* and *translational acceleration*. For each particle of a rotating object, the translational velocity and translational acceleration lay on the plane that is perpendicular to the rotation axis and contains the considered particle.

Figure 4

ROLLING (*an example of* ***general*** *or* ***composed rotation)***

-Theorem: ***The angular velocity (ω) of rotation around an axis passing by a given point on a body is equal to the angular velocity of rotation around a parallel axis passing by any other point on the body***.
Proof: Consider a body rotated CCW by angle $\boldsymbol{\Delta\theta}$ around an axis perpendicular to the plane of figure and passing by central point *'**c**'* (fig.5) during the interval of time ***Δt;*** i.e. *rotating at angular velocity* $\omega = \frac{\Delta\theta}{\Delta t}$**.**
After *rotation by* $\Delta\theta$*, the two points **a, b** on the body are shifted to **a', b'** respectively*. For an observer standing at *point "**a**" , during the same time **Δt** , the **line ab** is rotated anticlockwise by the angle* $\Delta\theta$**,** too.



*So, this observer would report a CCW **rotation around point "a"** with angular velocity* $\boldsymbol{\omega = \frac{\Delta\theta}{\Delta t}}$*. For an observer at point "**b**", the **line ba** is rotated anticlockwise by the angle **Δθ** during the time interval **Δt.** So, he would report a rotation around point "**b**" with the same angular velocity* $\boldsymbol{\omega = \frac{\Delta\theta}{\Delta t}}$**.** One, may figure out that by following the same logic the result is the same for any location of " *a point* " on object.

Figure 5

- Now, let's consider the ***uniform motion*** of a <u>*wheel rolling on the road*</u>. It is a ***composed rotation - translation*** motion[1]. While a point on the rim achieves a *full revolution around the center*, the rotation axis passing by *wheel center* travels a distance equal to wheel *circumference* (see fig.6). One assumes a *rotation without slide* and *a kind of static friction* between the wheel and road surface at each instant. During a complete revolution, i.e. during time " **T** " **,** the center of wheel has traveled the distance **2πR.** So, it comes out that the ***speed*** of the wheel center (which is *equal to speed of central axis of rotation*) ***versus ground*** is

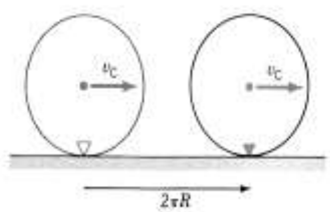


Figure 6

$$\upsilon_{trans-c} = \frac{2\pi R}{T} = \omega * R \qquad (18)$$

ω is the ***angular velocity*** of wheel rotation.

One can decompose the **motion** of ***any point*** of wheel into ***two components***; a ***translation " t "*** defined by the ***translation*** of the ***center of mass*** (**CM**) of the wheel (*and axis of rotation*) and a ***rotation "r "*** around **CM.** Then, the net displacement vs Oxy (*tied to ground*) is

$$\vec{\Delta s} = \vec{\Delta s}_{t-CM} + \vec{\Delta s}_{r-CM} \qquad (19)$$

and the ***velocity*** of a wheel point is

$$\vec{\upsilon} = \frac{d\vec{s}}{dt} = \vec{\upsilon}_{t-CM} + \vec{\upsilon}_{r-CM} \qquad (20)$$

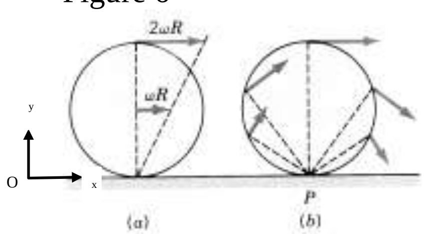


Figure 7

For a *point on the rim*, depending on its position, these ***two vectors*** may be aligned or not along the same direction but their magnitudes are equal

$$\upsilon_{t-CM} = \omega * R\_and\_\upsilon_{r-CM} = \omega * R \qquad (21)$$

-At the point of contact to the road "*P*", $\vec{v}_{t-CM}$ has ***opposite sense to*** $\vec{v}_{r-CM}$. As their sum is zero, this point has zero velocity and it is **momentarily at rest**. For the ***point at the top***, the **vectors** $\vec{v}_{t-CM}$ and $\vec{v}_{r-CM}$ have the ***same direction*** (fig.7.a). Their magnitudes sum up and its speed is ***υ = 2ωR.***
From another point of view, one may figure out that, at the shown instant, all wheel points are ***rotating*** at same angular velocity **ω** (*see the theorem above*) around the point of contact "*P*". So, their ***velocity vectors*** are perpendicular to the straight "*dashed line*" that goes from "*P*" to the considered point and their magnitudes are equal to **ω*distance to *P*** (*grey vectors in* fig.7.b). Therefore, the **magnitude of translational velocity,** i.e. the particle **speed increases with distance** from *P*- point. This effect can be easily observed on the spokes of the bike wheels; close to road they are easily seen (*low speed*) - at the top, they are all time blurred (*high speed*).

[1] Or a general rotation with respect to a frame tied to the earth

2] KINETIC ENERGY OF ROTATION AND THE MOMENT OF INERTIA

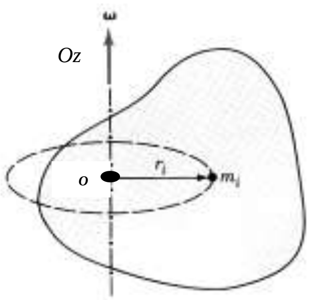


Figure 8

- The figure 8 shows a body rotating CCW at *constant angular velocity* $\vec{\omega}$ around a fixed axis. Its particles move at *different translational velocities* and the kinetic energy of body motion is the sum of their kinetic energies. Translational velocity of the particle with mass $\boldsymbol{m_i}$ at distance $\boldsymbol{r_i}$ ***from** the **axis*** of **rotation** is $\boldsymbol{\upsilon_i = \omega * r_i}$

and its kinetic energy is $$K_i = \frac{1}{2} m_i \upsilon_i^2 = \frac{1}{2} m_i r_i^2 \omega^2 \qquad (22)$$

Therefore, the kinetic energy of the whole body(*versus a frame with Oz along* $\vec{\omega}$) is

$$K = \sum_i K_i = \sum_i \frac{1}{2} m_i \upsilon_i^2 = \sum_i \frac{1}{2} m_i (\omega r_i)^2 = \frac{1}{2} \omega^2 * \sum_i m_i r_i^2 \qquad (23)$$

-The expression (23) can be written in form $$K = \frac{1}{2} I * \omega^2 \quad (24) \qquad \text{where} \qquad I = \sum_i m_i r_i^2 \quad (25)$$

Note: Expression (25) transforms into an integral when one calculates the magnitude of inertia moment for a rigid body.

The parameter ***I*** [*kg*m*^{2}] is called the ***moment of inertia*** of the body ***with respect to the axis of rotation*** "Oz". The numerical value of ***I*** depends on the way that the mass of the body is ***distributed spatially*** with respect to the ***considered axis***. A quick comparison of expression (24) with $\boldsymbol{K = 1/2\, m\upsilon^2}$ allows to figure out that, in a rotational motion, ***I***-parameter plays the same role as the mass in translational motion**.** So, it comes out that

The inertia moment is a measure of the "resistance" a body** (or a system of particles) **presents to the change of its rotational status of motion, i.e. to the change of its existing angular velocity.

-One may feel easily the ***opposition effect of inertia moment*** by trying to rotate a hammer. When one holds it by the wooden end, it is more difficult to rotate it (*bigger mass located at the free end, larger " r_i values" i.e. larger inertia moment*) than when holding it by the metallic end.

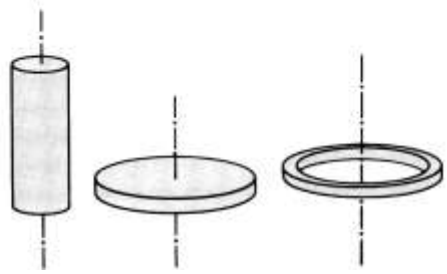


Figure 9

- Straight from the definition (25) it comes out that the same amount of mass located at a bigger distance from the axis produces larger moment of inertia. So, one may figure out that, for the ***same mass,*** the inertia moments versus the *central axis* of symmetry for a ring, a disk, and a cylinder (figure 9) are different and $\boldsymbol{I_{ring}} > \boldsymbol{I_{disk}} > \boldsymbol{I_{cylinder}}$. The **center of mass CM (***of an **object** or **system of particles***)** is a key parameter that helps to find ***I***-value for any position of rotation axis.

<u>CENTER OF MASS</u>

- When a wheel is rolling, **each rim point** participates simultaneously in two motions; a **translation** that is the same as the ***translation of the central point*** and a ***rotation around*** the ***central point***. This central point is the ***center of mass (CM)*** of the wheel. The wheel motion is an example of an object in *general rotation*. *The **CM** of a **system of particles (**which **may or may not constitute a rigid body) is <u>a point</u> (not necessary part of system or of body)** which **<u>motion</u> is a common characteristic for the motion of system of particles as a whole**.*

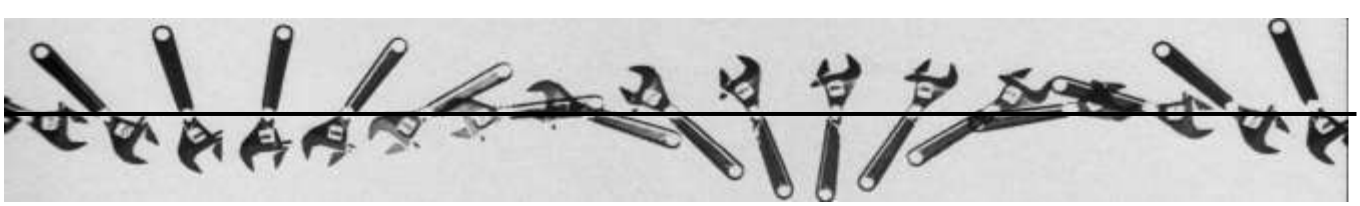


Figure 10 A wrench spinning over a horizontal frictionless surface. CM is moving at constant velocity.

In cases when one applies the model of ***single material point*** one assumes that the **whole mass** of the object of study **is located at "*a representative point*" which** is its **center of mass "CM"**.
If one has introduced a frame of reference Oxy and knows the locations (position vectors $\vec{r}_i$) of all particles that constitute the object (or the system of particles), the ***position of center of mass*** is found by the expression

$$\vec{r}_{CM} = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{M} \tag{26}$$

M is the mass of the object. The time derivative of (26) gives $$\vec{\upsilon}_{CM} = \frac{\sum_{i=1}^{n} m_i \vec{\upsilon}_i}{M} \tag{27}$$

By rewriting (27) as $M\vec{\upsilon}_{CM} = \sum_{i=1}^{n} m_i \vec{\upsilon}_i$ (28) and after another derivative versus time, one gets

Because the sum of internal forces of system of particles is zero (***due to third law***)

$$M\frac{d\vec{\upsilon}_{CM}}{dt} = \sum_{i=1}^{n} m_i \frac{d\vec{\upsilon}_i}{dt} \Rightarrow M*\vec{a}_{CM} = \sum_{i=1}^{n} m_i \vec{a}_i \Rightarrow M*\vec{a}_{CM} = \sum_{i=1}^{n} \vec{F}_i = \vec{F}_{NET\_EXT} \qquad \vec{F}_{Net\_EXT} = M*\vec{a}_{CM} \tag{29}$$

The relation (29) shows that the second law of Newton does apply on motion of ***CM.*** This means that ***CM moves in space*** as if ***a particle with mass M was*** placed at its location. For an ***<u>isolated system of particles</u>*** (*or an isolated single object*) $\vec{F}_{EXT.} = 0$. In this case, ***CM keeps its motion status " <u>at rest or uniform motion</u>"***.
The CM of wrench in fig. 10 moves at constant velocity (*along a straight line*) because the net exterior force (*weight plus normal*) on wrench is zero. If thrown at an angle to horizontal, the 2D motion of its CM would be following a parabola, like that of a single particle with mass M.

-Let's find the ***total kinetic energy*** of an object (*modelled as a system of particles, fig.11*) when it rotates around its CM(*frame O' tied to CM*) while CM moves versus frame O (*tied to earth*). The velocity of " ***i<u>th</u>*** " particle is

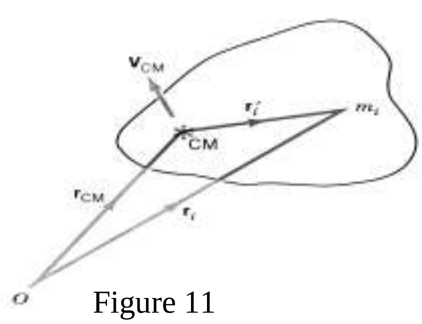


Figure 11

$$\vec{v}_i = \frac{d\vec{r}_i}{dt} = \frac{d(\vec{r}_{CM}+\vec{r}'_i)}{dt} \quad \text{and} \quad \vec{\upsilon}_i = \vec{\upsilon}_{CM} + \vec{\upsilon}'_i \tag{30}$$

where $\vec{\upsilon}'_i$ is its relative velocity versus frame O' (i.e. versus *CM).*

One can write its kinetic energy versus frame O as $K_i = \frac{1}{2} m_i \vec{\upsilon}_i * \vec{\upsilon}_i$ (31)

$$\text{and} \quad \vec{\upsilon}_i * \vec{\upsilon}_i = (\vec{\upsilon}_{CM} + \vec{\upsilon}'_i) * (\vec{\upsilon}_{CM} + \vec{\upsilon}'_i) = \upsilon_{CM}^2 + \upsilon_i'^2 + 2\vec{\upsilon}_{CM}\vec{\upsilon}'_i \tag{32}$$

The kinetic energy of the body versus O frame is equal to the sum of kinetic energies of all its particles. So,

$$K = \sum_i K_i = \sum_i \frac{1}{2} m_i \vec{\upsilon}_i * \vec{\upsilon}_i = \sum_i \frac{1}{2} m_i (\upsilon_{CM}^2 + \upsilon_i'^2 + 2\vec{\upsilon}_{CM}\vec{\upsilon}'_i) = \frac{1}{2}\upsilon_{CM}^2 \sum_i m_i + \frac{1}{2}\sum_i m_i \upsilon_i'^2 + \vec{\upsilon}_{CM}\sum_i m_i \vec{\upsilon}'_i$$

By using relation (28), one gets $\sum_i m_i \overrightarrow{v'}_i = M\vec{v}'_{CM} = 0$ because $\vec{v}'_{CM}$ is the velocity of CM versus CM (*a given point cannot move with respect to itself*) and the last term at K expression disappears.

Thus, $$K = \frac{1}{2}\upsilon_{CM}^2 \sum_i m_i + \frac{1}{2}\sum_i m_i \upsilon_i'^2 = \frac{1}{2}\upsilon_{CM}^2 M + K' = K_{CM} + K' = K_{CM} + K_{rot} \qquad (33)$$

where $K_{CM} = \frac{1}{2}M\upsilon_{CM}^2$ is the kinetic energy of CM motion versus fixed frame O.

and $$K_{rot} = \frac{1}{2}\sum_i m_i v_i'^2 = \frac{1}{2}\sum_i m_i(\omega r'_i)^2 = \frac{1}{2}\omega^2 \sum_i m_i r_i'^2 = \frac{1}{2}\omega^2 * I_{CM} = \frac{1}{2}I_{CM}\omega^2$$

is the kinetic energy of ***rotation around CM.*** $\boldsymbol{I}_{CM}$ is the inertia moment of object vs. its center of mass.

The ***kinetic energy*** of a body(*or system of particles*) in ***general rotational motion*** is the ***sum of kinetic energy of its CM*** *and its* ***rotational kinetic energy versus the CM***.

THE PARALLEL AXIS THEOREM

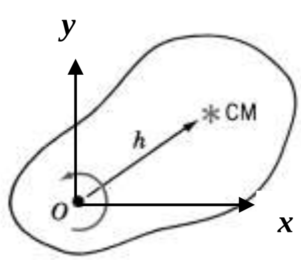


Figure 12

-This theorem *relates the moment of inertia* ***I*** *about* ***any axis*** *of rotation to the* ***moment of inertia*** $\boldsymbol{I_{CM}}$ *about a* ***parallel*** *axis* ***passing through CM***. Let's consider an axis **Oz** *perpendicular* to the section of object (shown in figure) and passing by the point O at *distance* ***h*** from its ***CM***. Suppose that the object rotates CCW around Oz at angular velocity $\boldsymbol{\omega}$. As explained previously, each point of the body is rotating at same angular velocity $\boldsymbol{\omega}$ around a parallel axis passing by **CM** point, too. Note that ***a body in motion has a given numerical value of kinetic energy (versus a given fixed frame O) no matter the method used to calculate it***. We are going to calculate it in two ways; **a)** by using the *expression* (33) for a general (*or composed*) rotation and **b)** by using the *expression* (24) for a pure rotation around Oz.

a) ***The kinetic energy of body rotating around axis Oz passing by O, can be calculated as the sum of the kinetic energy of its CM motion plus its kinetic energy due to rotation around CM***(see 33)

$$K = K_{CM} + K_{rot\_vs\_CM} = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}I_{CM}\omega^2 \qquad (34)$$

In our case (see fig. 12) the point CM is rotating around point O at angular velocity $\boldsymbol{\omega}$. So, its translational or **linear velocity** is

$$\upsilon_{CM} = \omega * h \qquad (35)$$

By substituting (35) at (34) we get $$K = \frac{1}{2}M\omega^2 h^2 + \frac{1}{2}I_{CM}\omega^2 = \frac{1}{2}(I_{CM} + M * h^2)\omega^2 \qquad (36)$$

b) By using expression (24) for the axis Oz passing by O we have $$K = \frac{1}{2}I_{Oz}\omega^2 \qquad (37)$$

As the results of expressions (36-37) must be the same, we get $$I_{Oz} = I_{CM} + M * h^2 \qquad (38)$$

- The **inertia moments** for axes passing through CM for some basic ***body shapes*** are given in tables. One may calculate ***I***-values for *parallel axes* passing by different positions of the body by use of relation (38).

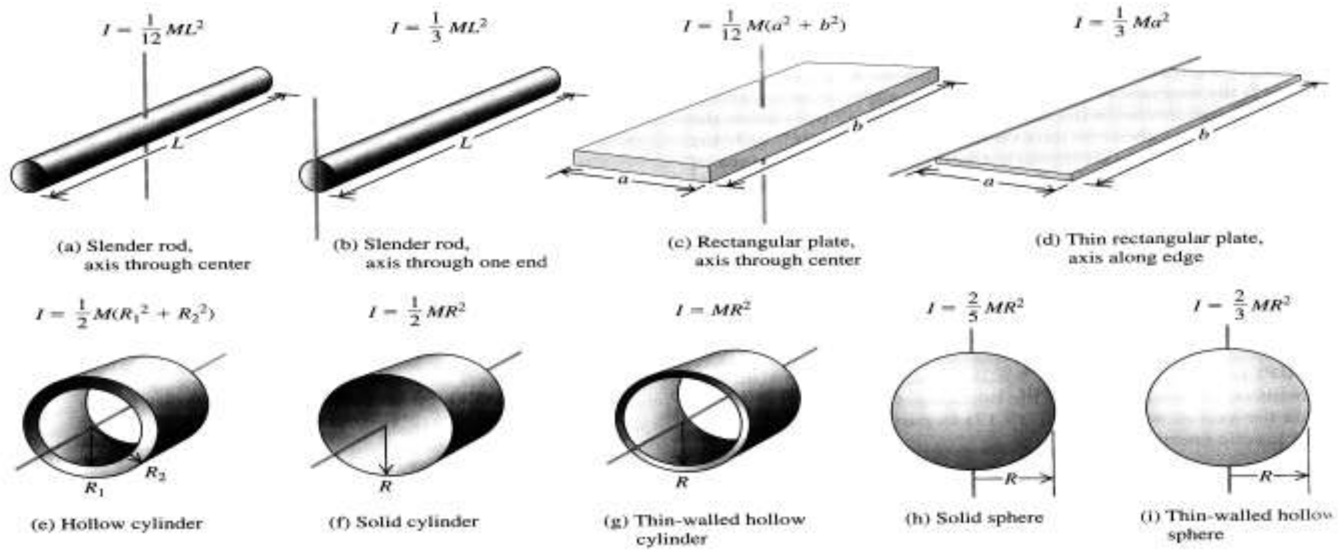


Figure 13

3] CONSERVATION OF MECHANICAL ENERGY IN A GENERAL ROTATION

- The mechanical energy is conserved in situations where there is **no external work done over the system**. This is valid no matter the type of motion a given system is doing. Let's apply this principle for a sphere with mass ***M*** and radius ***R*** that rolls *without slipping* on an inclined plane (fig.14).
In ***rolling*** *motion of a* ***rigid*** *body*, f = f_{static} and $\boldsymbol{W_{f\text{-}static} = 0}$. Then, as $W_N = 0$, one get $W_{ext} = W_N + W_{f\text{-}static} = 0$ for the system *Earth&Sphere*. Thus, it comes out that *mechanical energy* E(H) for the sphere at initial height is equal to ***E(h)*** when the sphere is at *any level "h"*. Next, we will refer to the end of inclined plane where ***h=0*** and write the principle of mechanical energy conservation as follows ***E(H) = E(0)*** (39)
At H-level, there is *only potential energy*; ***E(H) = MgH*** (40)

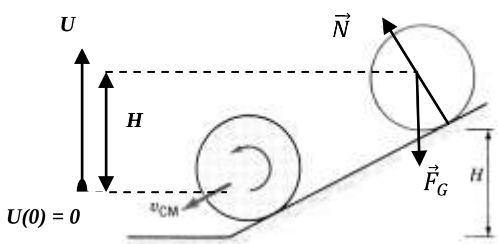


Figure 14

Notes: a) *The expression E(H) =U(H) = MgH means that we have fixed the 0-value for potential energy at the level of* ***CM*** *when the sphere is rolling on horizontal plane*.

b) Remember that, *the potential energy of sphere (MgH) belongs to the system sphere-earth and this means that the gravitation force is not an external force for this system.*

- At the end of inclined plane (h = 0) the sphere is participating in **two motions**, a **rotation around** its **CM** and a **translation with its CM**. So, its kinetic energy is (see 33) $K = \frac{1}{2}M\upsilon_{CM}^2 + \frac{1}{2}I_{CM}\omega^2$ (41)

As its potential energy is U(h=0) = 0 we get $E(0) = K(0) + U(0) = \frac{1}{2}M\upsilon_{CM}^2 + \frac{1}{2}I_{CM}\omega^2$ (42)

Then, by using expression (40) $E(H) = E(0) \Rightarrow\Rightarrow MgH = \frac{1}{2}M\upsilon_{CM}^2 + \frac{1}{2}I_{CM}\omega^2$ (43)

As $\upsilon_{CM} = \omega * R$ we get $MgH = \frac{1}{2}M\upsilon_{CM}^2 + \frac{1}{2}I_{CM}\frac{\upsilon_{CM}^2}{R^2}$ and $\upsilon_{CM} = \sqrt{\left(\frac{2MgH}{M + I_{CM}/R^2}\right)}$ (44)

Note: If there is a moving system of objects tied by a rope that pass over a pulley without sliding on it, one must take into account the kinetic energy of rotating pulley (**1/2***$\boldsymbol{I_{CM\_pulley}}$ ***$\boldsymbol{\omega^2}$**) *during the motion of system.*