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LECTURE 12

INTRODUCTION

-Until now, we were referring to the ***model of object as a single material point*** (i.e. a particle with object's mass) and have studied the motion of object by analyzing the motion of this particle in space. This model is valid if **all points of object** have the **same vector of displacement**. This condition is fulfilled in case of translational motion and even for rotational motion if the distance of object from the axe of rotation is much larger than the object dimensions (*in this case, **the distances of all object points from the rotation axe are <u>practically</u> equal)***.

In the following we will refer to the rotational motion around an **axis passing through** the **object** or **outside object but close** to it. In these circumstances the **displacement vector is different for each point of object.** Therefore one **<u>cannot</u>** model the object motion as **a single material point** motion. In those situations, one has to model the whole object as ***a set of particles*** *that* ***rotate simultaneously together*** *around the same axis of rotation*.

- The ***rigid body*** is the model of an object which **shape** remains **unchanged** during its motion. We will consider the ***<u>rotation of a rigid body</u>*** *around* a ***<u>fixed axis</u>*** in an ***<u>inertial frame Oxyz</u>***. This axis of rotation does not move versus the frame of reference while the body rotates around it (fig.1). *In general, one places Oz axe along the axis of rotation.*

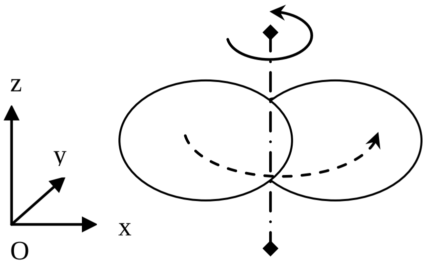


Figure 1

In a *pure rotational motion* all body ***particles move on circular paths centered on the axis of rotation***. ***Spinning*** is a *pure rotation* around an *axis passing through* the center of mass (CM) of the object.
In a *general rotational motion,* the *<u>axis of rotation can move with respect to the reference frame</u>* but it remains always *<u>fixed with respect to the body</u>*. One studies the *general rotation* as a *<u>composed motion</u>* composed by a pure rotation and a translation motion.

1] ROTATIONAL KINEMATICS IN A PURE ROTATIONAL MOTION

-How to describe the motion of a rigid body particles during a *pure rotational motion*? At first, one notes that during such type of rotation, **all particles** of object **rotate** by the **same angle** around **same rotation axis** even though their **displacement vectors are different**. Let's consider the pure CCW rotation of a body around an axe passing through it (*by point O and perpendicular to page in figure 2*) during a given interval of time.

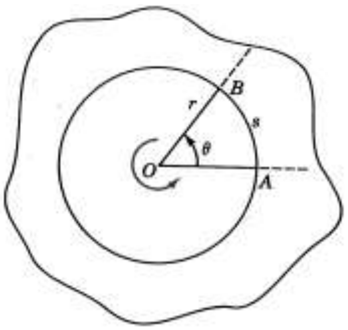


Figure 2
A section of body perpendicular to the axis of rotation

During this interval of time, all the particles on the direction OA get rotated by the same angle θ. Meanwhile, the travelled distance for each of them depends on the distance from the axe (i.e. point O in figure). For point *A at distance " r "* from the axis of rotation, the length of *travelled distance* "s" is

$$s = r * \theta \qquad (1)$$

(***θ*** *in* ***radian,*** counted versus a fixed reference direction "*say OA*"***, taken "+" for CCW***). Assuming that the body rotates by the angle ***Δθ*** during the short interval of time ***Δt,*** one defines its ***average angular velocity*** as

$$\omega_{av} = \frac{\Delta\theta}{\Delta t} = \frac{\theta_f - \theta_i}{t_f - t_i} \qquad (2)$$

If the rotation rate changes in time one uses the ***instantaneous angular velocity***

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \qquad (3)$$

The *angular velocities* (ω_{av}, ω) give the number of radians covered in unit of time; their SI unit is [***rad / sec***].

- In fact, the **angular velocity is a vector** that informs about the *sense of rotation*, too. Its **direction** is defined by the **right hand rule;** if one curls the *four fingers of right hand* following the rotation sense of object, the thumb gives the direction of vector $\vec{\omega}$. The vector $\vec{\omega}$ is **placed on the rotation axis**. Like in all physics studies, one should start by defining a reference frame of axes. For ***pure rotations,*** one places **Oz** axis ***along the rotation axis*** and ***Ox, Oy perpendicular to Oz.*** Then, if when *looking down the Oz axis*, the object rotates ***counter clockwise*** *"CCW"* one considers the rotation to be ***positive*** (case of fig.3) and the vector $\vec{\omega}$ is aligned along positive direction of Oz. Otherwise, $\vec{\omega}$ is aligned in opposite direction to Oz, the rotation is ***clockwise*** "CW" and the rotation is considered to be ***negative***.
The expressions (2, 3) give the component(scalar) of vector $\vec{\omega}$ on Oz axis.

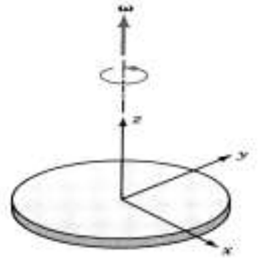


Figure 3

- If the *magnitude of angular velocity* is constant ($\omega = c^{\underline{te}}$), one deals with an *uniform rotation*. One has labelled as a ***period " T "*** the *time it takes for a full revolution of object* in an ***uniform rotation***. From the expression (2) one can get $\Delta t = \Delta\theta / \omega_{av} = \Delta\theta / \omega$. As for one revolution $\Delta t = T$ and $\Delta\theta = 2\pi$, it comes out that

$$T = \frac{2\pi}{\omega} \qquad (4)$$

The frequency of rotations "***f ”*** [**Hz** or **sec**$^{-1}$] is the ***number of revolutions*** completed in ***one second***.
So, for uniform rotations, one gets $f = \frac{1}{T}$ (5) and $\omega\left[\frac{rad}{s}\right] = \frac{2\pi}{T[s]} = 2\pi f[Hz]$ (6)

-The *instantaneous speed* of a particle is equal to *magnitude of its instantaneous velocity* $\vec{v} = \frac{d\vec{s}}{dt}$ and it is defined by the magnitude of displacement "***ds***" for an infinitesimal time "***dt*** ". For a particle at distance " ***r*** " from rotation axis and an infinitesimal rotation by "***dθ*** " the relation (1) would give its *infinitesimal displacement magnitude* as

$$ds = r * \theta \qquad (7)$$

Then, the *instantaneous speed* of this particle is $v(r) = \frac{ds}{dr} = \frac{d(r\theta)}{dt} = r\frac{d\theta}{dt} = r * \omega$ (8)

Expression (8) shows that the ***speed of object’s particles*** increases with the distance "***r***" from the axis of rotation. Also, all particles at the same distance " ***r*** " from rotation axis move at same *speed*.

- For situations where the ***angular velocity*** is not constant (i.e. ***non-uniform rotations***), one has to use the ***angular acceleration***. One has defined the ***average angular acceleration*** as $\alpha_{av}\left[\frac{rad}{s^2}\right] = \frac{\Delta\omega}{\Delta t}$ (9)

and the ***instantaneous angular acceleration*** as $\alpha\left[\frac{rad}{s^2}\right] = lim_{\Delta t\to 0}\frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$ (10)

Actually, the **angular acceleration** of a rotation **is a vector $\vec{\alpha}$ that lays on the rotation axis Oz,** too**.**
So, it has only one component " $\alpha_z = \alpha$ ". If $\boldsymbol{\alpha_z}$ has the ***same sign*** (+ or -) as $\boldsymbol{\omega_z}$, this means that $\vec{\alpha}$ has the **same sense** as $\vec{\omega}$ and the rotation is ***speeding up***; if they have **opposite sign** the rotation is ***slowing down***.

- For ***constant angular acceleration,*** ***"α" is constant*** *and* one can rewrite the expression (9) as follows

$\Delta\omega = \alpha_{av} * \Delta t \equiv \alpha * \Delta t.$ Next, by taking $t_i = 0$; $\Delta t = t_f - t_i = t_f \equiv t$ one gets $\boldsymbol{\omega - \omega_0 = \alpha * t}$ (11)

Relation (11) has the same mathematical form as that of translational velocity in a 1D motion at **constant acceleration**. By using the same area technique as in 1D *kinematics* (see lecture 4), one will find out that
the ***angle of rotation*** is expressed as $\theta = \theta_0 + \omega * t + \frac{1}{2}\alpha * t^2$ (12)
and the ***angular velocity*** is expressed as $\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$ (13)

So, for a motion at **_CONSTANT_**

Translational Acceleration (a = c^{te})	***Angular acceleration (α = c^{te})***
$v = v_0 + a * t$	$\omega = \omega_0 + \alpha * t$
$x - x_0 = v_0 * t + \frac{1}{2} * t^2$	$\theta - \theta_0 = \omega_0 * t + \frac{1}{2}\alpha * t^2$
$v^2 = v_0^2 + 2a(x - x_0)$	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$
$x - x_0 = \frac{v_0 + v}{2} * t$	$\theta - \theta_0 = \frac{\omega_0 + \omega}{2} * t$

-If the object rotates **uniformly** (i.e. $\omega = c^{te}$; $\alpha = 0$), any particle at distance " ***r*** " from rotation axe moves at same ***constant speed v(r)*** given by (8). As we know, a particle that rotates uniformly on a circular path with radius " ***r*** " has a centripetal (*or radial*) acceleration $a_{c-r} = \frac{v_r^2}{r}$ and by using the expression (8), one will get

$$a_{c-r} = \frac{v_r^2}{r} = \frac{\omega^2 * r^2}{r} = \omega^2 * r \tag{14}$$

This quantity (*centripetal* ***or radial) acceleration*** is a ***vector $\vec{a}_{c-r}$ directed versus the center of rotation*** (O-point***)***. Remember that, always, the vector $\vec{a}$ is aligned the same way as the vector $\Delta\vec{v}$. This means that, in a ***uniform rotation***, even though the *speed is constant* there is a *change of velocity vector* and it is *directed versus the center of rotation*.

In an ***accelerated circular motion***, the magnitude of *angular velocity **ω** changes* in time (see expr. 11). Then, the relation (8) tells that the magnitude of *velocity vector $\vec{v}$*, will change, too. One may find easily that, in this case, there is a component of velocity change $\Delta\vec{v}$ directed *along the tangent* to the circular path and it produces a *tangential acceleration* a_{t-r} equal to the derivative of the particle speed (i.e. the derivative of expr. *8*)

$$a_{t-r} = \frac{dv}{dt} = \frac{d(\omega * r)}{dt} = \frac{d\omega}{dt} * r = \alpha * r \tag{15}$$

So, in a ***non-uniform*** rotation, a particle at distance " *r* " moves at a ***net (****translational****) acceleration***

$$\vec{a}_r = \vec{a}_{c-r} + \vec{a}_{t-r} \tag{16}$$

The ***centripetal and tangential accelerations*** lay on the plane of particle rotation and are ***perpendicular to each other***. So, the vector of ***net acceleration*** lies on the same plane (fig 4) and has magnitude

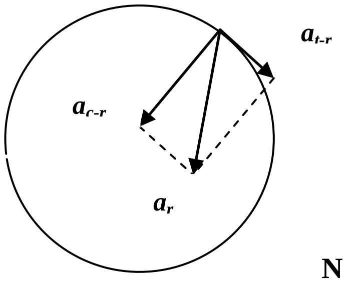


Figure 4

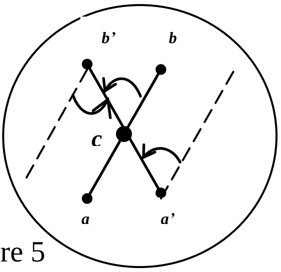
$$a_r = \sqrt[2]{a_{c-r}^2 + a_{t-r}^2} \tag{17}$$

NOTE: One uses the term *angular velocity* and *angular acceleration* to distinguish them from *translational velocity* and *translational acceleration*. For each particle of a rotating object, the translational velocity and translational acceleration lay on the plane that is perpendicular to the rotation axis and contains the considered particle.

ROLLING (**an example of **general** or **composed rotation)

-Theorem: ***The angular velocity "ω" of rotation around an axis passing by a given point of a body is equal to the angular velocity of rotation around a parallel axis passing by any other point of the body.***

Proof: Consider a body rotated CCW by angle $\Delta\boldsymbol{\theta}$ around an axis perpendicular to the plane of figure and passing by central point ***'c'*** (see fig.5) during the interval of time ***Δt;*** i.e. *rotating at angular velocity* $\omega = \frac{\Delta\theta}{\Delta t}$ **.** After *rotation by* $\Delta\theta$, *the two points* ***a, b*** *of the body are shifted to* ***a', b'*** *respectively.* For an observer standing at *point "**a**", during the same time* ***Δt***, *the* **line ab** *is rotated anticlockwise by the same angle* $\Delta\theta$.



So, this observer would report a CCW **rotation around point "a" at same** *angular velocity* $\boldsymbol{\omega} = \frac{\Delta\boldsymbol{\theta}}{\Delta\boldsymbol{t}}$**.** *For an observer at point "**b**", the* **line ba** *is rotated anticlockwise by the angle **Δθ** during the time interval **Δt**, **too.** So, he would report a rotation around point "**b**" at the same angular velocity* $\boldsymbol{\omega} = \frac{\Delta\boldsymbol{\theta}}{\Delta\boldsymbol{t}}$**.** One, may figure out that by following the same logic the result is the same for any location of " *a point* " of object.

Figure 5

- Now, let's consider the ***uniform motion*** of a <u>*wheel rolling on the road*</u>. It is a ***composed rotation - translation*** motion[1]. While a point on the rim achieves a *full revolution around the center,* the rotation axis passing by *wheel center* travels a distance equal to wheel *circumference* (see fig.6). One assumes a *rotation without slide* and *a kind of static friction* between the wheel and road surface at each instant. During a complete revolution, i.e. during time " ***T*** " **,** the center of wheel has traveled the distance ***2πR.*** So, it comes out that the ***speed*** of the ***wheel center vs. ground*** is

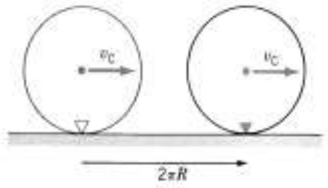


Figure 6

(also, it is the *speed of central axis of rotation*)

$$v_{trans-c} = \frac{2\pi R}{T} = \omega * R \tag{18}$$

where **ω** is the ***angular velocity*** of wheel rotation.

One can decompose the **motion** of ***any point*** of wheel into ***two components***; a ***translation "t"*** defined by the ***translation*** of the ***center of mass*** (**CM**) of the wheel (*and axis of rotation*) and a ***rotation "r "*** around **CM.** Then, the net displacement vs Oxy (*tied to ground*) is

$$\Delta\vec{s} = \Delta\vec{s}_{t-CM} + \Delta\vec{s}_{r-CM} \tag{19}$$

and the ***velocity*** of a wheel point is

$$\vec{v} = \frac{d\vec{s}}{dt} = \vec{v}_{t-CM} + \vec{v}_{r-CM} \tag{20}$$

For a *point on the rim,* depending on its position, these ***two vectors*** may be aligned or not along the same direction but their magnitudes are equal

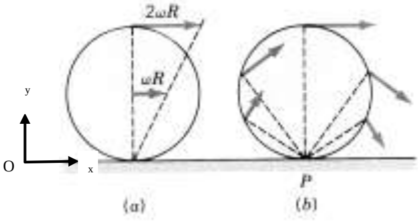


Figure 7

$$v_{t-CM} = \omega * R \quad \text{and} \quad v_{r-CM} = \omega * R \tag{21}$$

-At the point of contact to the road "*P*", $\vec{v}_{t-CM}$ has ***opposite sense to*** $\vec{v}_{r-CM}$. As their sum is zero, this point has zero velocity and it is **momentarily at rest**. For the ***point at the top***, the ***vectors*** $\vec{v}_{t-CM}$ and $\vec{v}_{r-CM}$ have the ***same direction*** (fig.7.a). Their magnitudes sum up and the speed of this point is ***υ = 2ωR.***

From another point of view, one may figure out that, at the shown instant, all wheel points are ***rotating*** at same angular velocity **ω** (*see the theorem above*) around the point of contact "*P*". So, their ***velocity vectors*** (*grey vectors in* fig.7.b) are perpendicular to the straight "*dashed line*" that goes from "*P*" to the considered point and their magnitudes are equal to **ω*distance to *P***. Therefore, the **magnitude of translational velocity,** i.e. the particle **speed increases with distance** from *P*- point. This effect can be easily observed on the spokes of the bike wheels; close to road they are easily seen (*low speed*) - at the top, they are all time blurred (*high speed*).

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[1] Or a general rotation with respect to a frame tied to the earth

2] KINETIC ENERGY OF ROTATION AND THE MOMENT OF INERTIA

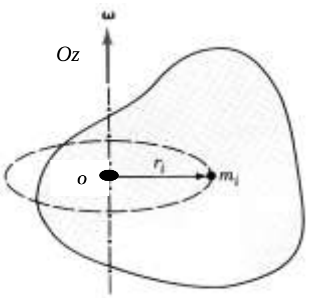


Figure 8

- The figure 8 shows a body rotating CCW at *constant angular velocity* $\vec{\omega}$ around a fixed axis. Its particles move at *different translational speeds* and the kinetic energy of body motion is the sum of their kinetic energies. Translational speed of the particle with mass ***m***$_i$ at distance ***r***$_i$ ***from*** *the* ***axis*** of ***rotation*** is $v_i = \omega * r_i$ and its kinetic energy is

$$K_i = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i \omega^2 r_i^2 \qquad (22)$$

Therefore, the kinetic energy of the whole body(*versus a frame with Oz along* $\vec{\omega}$) is

$$K = \sum_i K_i = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i \omega^2 r_i^2 = \frac{1}{2} \omega^2 \sum_i m_i r_i^2 \qquad (23)$$

-The expression (23) can be written in form $K = \frac{1}{2} I * \omega^2$ (24) where $I = \sum_i m_i r_i^2$ (25)

Note: Expression (25) transforms into an integral when one calculates it for a rigid body.

The parameter ***I*** [*kg*m*^{2}] is called the ***moment of inertia*** of the body ***with respect to the axis of rotation*** "Oz". Note that the numerical value of ***I*** depends on the way that the mass of the body is ***distributed spatially*** with respect to the ***considered axis***. A quick comparison of expression (24) with $K = 1/2\ mv^2$ allows to figure out that, in a rotational motion, ***I***-parameter plays a similar role as the mass "m" in a translational motion**.** So, it comes out that:

The inertia moment is a measure of the "resistance" that a body *(or a system of particles)* ***presents to the change of its rotational status of motion, i.e. to the change of its existing angular velocity.***

-One may feel easily the ***opposition effect of inertia moment*** by trying to rotate a hammer. It is more difficult to rotate the hammer if one holds it by the wooden end (*bigger mass located at the free end with larger "* r_{i-} *values" i.e. larger inertia moment*) than when holding it by the metallic end.

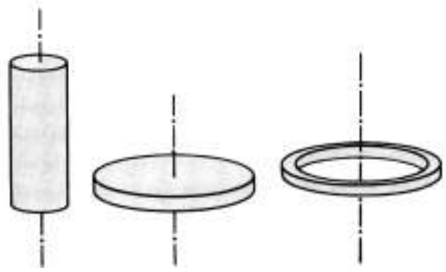


Figure 9

- Straight from the definition (25) it comes out that the same amount of mass located at a bigger distance from the axis of rotation produces larger moment of inertia. So, one may figure out that, for the ***same mass,*** the inertia moments versus the *central axis* of symmetry for a ring, a disk, and a cylinder (figure 9) are different and $I_{ring} > I_{disk} > I_{cylinder}$.
The **center of mass CM (***of an* ***object*** *or* ***system of particles*)** is a key parameter that helps to find ***I***-value for any position of rotation axis.

CENTER OF MASS

- When a wheel is rolling, **each rim point** participates simultaneously in two motions; a **translation** that is the same as the ***translation of the central point*** and a ***rotation around*** the ***central point***. This central point is the ***center of mass*** **(CM)** of the wheel. The wheel motion is an example of an object in *general rotation (or rotation-translation motion).*

The **CM** *of a* ***system of particles*** *(which* ***may or may not constitute a rigid body) is a point (not necessary one of the body or the system particles)*** *which* ***motion is a common characteristic for the motion of the whole system of particles.***

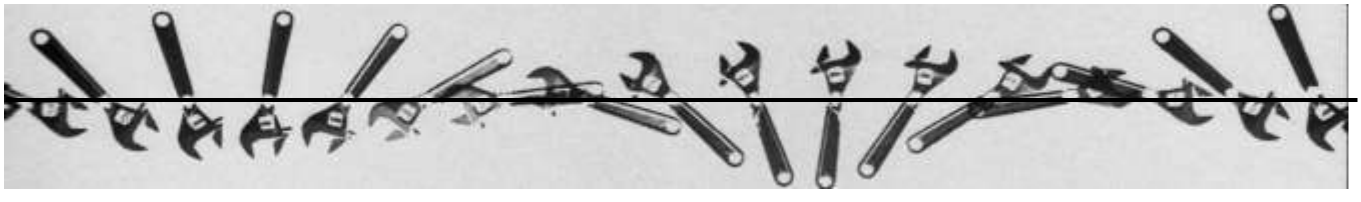


Figure 10 A wrench spinning while sliding uniformly over a horizontal frictionless surface.
CM is moving along the straight line at constant velocity.

Note that in cases when one can apply the model of ***single material point*** one assumes that the **whole mass** of the object of study **is located at "*a representative point*" which** is the **center of mass "CM"** of the object.
If one has introduced a frame of reference Oxy and knows the locations (position vectors $\vec{r}_i$) of all the particles that constitute the system of particles (or the object), one can find the ***position of center of mass*** by the expression

$$\vec{r}_{CM} = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{M} \qquad (26)$$

M is the mass of the all particles. The time derivative of (26) gives $\vec{v}_{CM} = \frac{\sum_{i=1}^{n} m_i \vec{v}_i}{M}$ (27)

By rewriting (27) as $M\vec{v}_{CM} = \sum_{i=1}^{n} m_i \vec{v_i}$ (28) and after another derivative versus time, one gets

$$M\frac{d\vec{v}_{CM}}{dt} = \sum_{i=1}^{n} m_i \frac{d\vec{v}_i}{dt} \to M\vec{a}_{CM} = \sum_{i=1}^{n} m_i \vec{a}_i \to M\vec{a}_{CM} = \sum_{i=1}^{n} \vec{F}_i = \vec{F}_{Net\_ext} \quad \text{i.e.} \quad \vec{F}_{Net\_Ext} = M\vec{a}_{CM} \qquad (29)$$

Because the sum of internal forces of the system of particles is zero (***due to third law***)

The relation (29) shows that the second law of Newton does apply on motion of ***CM.*** This means that ***CM moves in space*** as if ***a particle with mass M was*** placed at its location. For an **<u>*isolated system of particles*</u>** *(or an isolated single object)* $\vec{F}_{Net\_eXT.} = 0$ and in this case, ***CM keeps its motion status " <u>at rest or uniform motion</u> "***.
The CM of wrench in fig. 10 moves at constant velocity (along a straight line) because the net exterior force (weight plus normal) on wrench is zero. If thrown at an angle to horizontal, the 2D motion of its CM would be following a parabola, like that of a single particle with mass M.

-Now, let's find the ***total kinetic energy*** of an object (*modelled as a system of particles, see fig.11*) when it rotates around its CM (a *frame O' is tied to CM*) while CM moves versus frame O (*tied to earth*). The velocity of " ***i***th " particle is

$$\vec{v}_i = \frac{d\vec{r}_i}{dt} = \frac{d(\vec{r}_{CM} + \vec{r'}_i)}{dt} = \frac{d(\vec{r}_{CM})}{dt} + \frac{d(\vec{r'}_i)}{dt} \quad \text{and} \quad \vec{v}_i = \vec{v}_{CM} + \vec{v}'_i \qquad (30)$$

where $\vec{\upsilon}'_i$ is its relative velocity versus frame O' (i.e. versus *CM).*

Its kinetic energy versus frame O is $K_i = \frac{1}{2} m_i \vec{v_i} * \vec{v}_i$ (31)

and

$$\vec{v}_i * \vec{v}_i = (\vec{v}_{CM} + \vec{v}'_i) * (\vec{v}_{CM} + \vec{v}'_i) = v_{CM}^2 + v'^2_i + 2\vec{v}_{CM} * \vec{v}'_i \qquad (32)$$

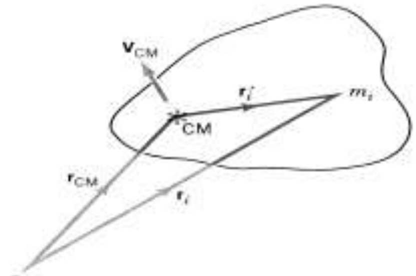


Figure 11

So, $K_i = \frac{1}{2} m_i (v_{CM}^2 + v'^2_i + 2\vec{v}_{CM} * \vec{v}'_i)$ (33)

Then, the kinetic energy of the body versus O frame is

$$K = \sum_i K_i = \sum_i \frac{1}{2} m_i \left(v_{CM}^2 + v'^2_i + 2\vec{v}_{CM} * \vec{v}'_i\right) = \frac{1}{2} v_{CM}^2 \sum_i m_i + \frac{1}{2} \sum_i m_i v'^2_i + \vec{v}_{CM} \sum_i m_i \vec{v}'_i \qquad (34)$$

By using relation (28) in frame O´, one gets $\sum_i m_i \overrightarrow{v'}_i = M\vec{v}'_{CM} = 0$ because $\vec{v}'_{CM}$ is the velocity of CM versus CM (*a given point cannot move with respect to itself*) and the last term at expression (34) disappears.

So $$\boldsymbol{K} = \frac{1}{2} v_{CM}^2 \sum_i m_i + \frac{1}{2} \sum_i m_i v'^2_i = \frac{1}{2} v_{CM}^2 * M + K' = \boldsymbol{K_{CM}} + \boldsymbol{K_{rot}} \qquad (35)$$

where $\boldsymbol{K_{CM}} = \frac{\boldsymbol{1}}{\boldsymbol{2}} \boldsymbol{M} * \boldsymbol{v_{CM}^2}$ is the kinetic energy of CM motion versus fixed frame O.

and $\boldsymbol{K_{rot}} = \frac{1}{2} \sum_i m_i v'^2_i = \frac{1}{2} \sum_i m_i (\omega r'_i)^2 = \frac{1}{2} \omega^2 \sum_i m_i r'^2_i = \frac{1}{2} \omega^2 * I_{CM} = \frac{\boldsymbol{1}}{\boldsymbol{2}} \boldsymbol{I_{CM}} * \boldsymbol{\omega^2}$ is the kinetic energy of ***rotation around CM. I***$_{CM}$ is the inertia moment of object vs. its center of mass.

The ***kinetic energy*** of a body(*or system of particles*) in a ***general rotational motion*** is the ***sum of translational kinetic energy of its CM*** *and its* ***rotational kinetic energy versus the CM***.

THE PARALLEL AXIS THEOREM

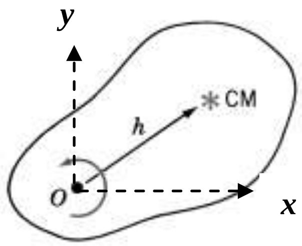


Figure 12

-This theorem *relates the moment of inertia* ***I*** *about* ***any axis*** *of rotation to the* ***moment of inertia*** $\boldsymbol{I_{CM}}$ *about a* ***parallel*** *axis* ***passing through CM***. Let's consider an axis **Oz** *perpendicular* to the section of object (shown in figure) and passing by the point O at *distance* ***h*** from its ***CM***. Suppose that the object rotates CCW around Oz at angular velocity $\boldsymbol{\omega}$. As explained previously, each point of the body is rotating at same angular velocity $\boldsymbol{\omega}$ around a parallel axis passing by **CM** point, too. Note that ***a body in motion has a given numerical value of kinetic energy (versus a given fixed frame O) no matter the method used to calculate it***. We are going to calculate it in two ways;

a) by using the *expression* (35) for a general (*or composed*) rotation and

b) by using the *expression* (24) for a pure rotation around Oz.

a) ***The kinetic energy of body rotating around axis Oz passing by O, can be calculated as the sum of the kinetic energy of its CM motion plus its kinetic energy due to rotation around CM***(see 35)

$$K = K_{CM} + K_{rot\_vs\_CM} = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}I_{CM}\omega^2 \tag{36}$$

In our case (see fig. 12) the point CM is rotating around point O at angular velocity $\boldsymbol{\omega}$. So, its translational speed in frame O is

$$\upsilon_{CM} = \omega * h \tag{37}$$

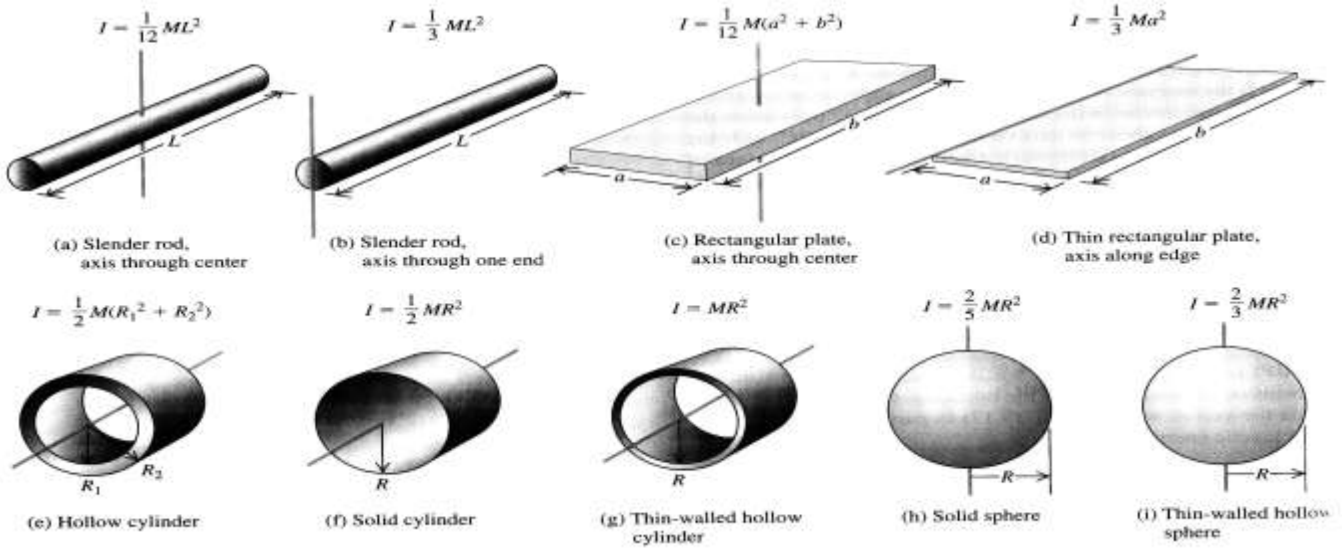
By substituting (37) at (36) we get $K = \frac{1}{2}M\omega^2h^2 + \frac{1}{2}I_{CM}\omega^2 = \frac{1}{2}(I_{CM} + Mh^2)\omega^2$ (38)

b) By using expression (24) for the axis Oz passing by O we have $K = \frac{1}{2}I_{Oz}\omega^2$ (39)

As the results of expressions (36-37) must be the same, we get $I_{Oz} = I_{CM} + M * h^2$ (40)

- The **inertia moments** for axes passing through CM for some basic ***body shapes*** are given in tables.

One may calculate ***I***-values for *parallel axes* passing by different positions of the body by use of relation (40).



(a) Slender rod, axis through center

(b) Slender rod, axis through one end

(c) Rectangular plate, axis through center

(d) Thin rectangular plate, axis along edge

(e) Hollow cylinder

(f) Solid cylinder

(g) Thin-walled hollow cylinder

(h) Solid sphere

(i) Thin-walled hollow sphere

Figure 13

3] CONSERVATION OF MECHANICAL ENERGY IN A ROTATION-TRANSLATION

- The mechanical energy is conserved in situations where there is **no external work done over the system**. This is valid no matter what is the type of motion in a system. Let's apply this principle for the case of a sphere with mass ***M*** and radius ***R*** that rolls *without sliding* down an inclined plane (fig.14).

In *a **rolling**[i] motion of a **rigid** body*, f = f_{static}. As the tail of f_{static} does not move, we get ***$W_{f\text{-}static} = 0$***. Then, as $W_N = 0$, one get $W_{ext} = W_N + W_{f\text{-}static} = 0$ for the system *Earth&Sphere*. Thus, it comes out that *mechanical energy* E(H) for the sphere at the initial height H is equal to the mechanical energy ***E(h)*** of the sphere at *any level "h"*. Next, we will refer to the end of inclined plane where ***h=0*** and write the principle of mechanical energy conservation as follows ***E(H) = E(0)*** (41)

At H-level, there is *only potential energy*; ***E(H) = MgH*** (42)

Notes: a) *The expression E(H) = MgH means that at height H all the energy is gravitational potential energy. One has fixed the 0-value for potential energy at the level of **CM** when the sphere is rolling on the horizontal plane.*

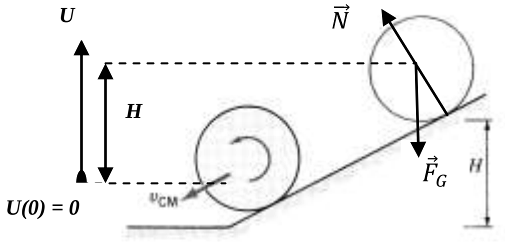


Figure 14

b) Remember that, *the potential energy of sphere (MgH) belongs to the system sphere-earth and the gravitation force is not an external force for this system.*

- At the end of inclined plane (h = 0) the sphere is participating in **two motions**, a **rotation around** its **CM** and a **translation with its CM**. So, its kinetic energy is (see 35) $K = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}I_{CM}\omega^2$ (43)

As its potential energy is U(h=0) = 0 we get $E(0) = K(0) + U(0) = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}I_{CM}\omega^2$ (44)

Then, by using expression (41) $\text{MgH} = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}I_{CM}\omega^2$ (45)

As $v_{CM} = \omega * R$ or $\omega = \frac{v_{CM}}{R}$ we get $\text{MgH} = \frac{1}{2}Mv_{CM}^2 + \frac{1}{2}I_{CM}\frac{v_{CM}^2}{R^2} = \frac{1}{2}(M + \frac{I_{CM}}{R^2})v_{CM}^2$

and $v_{CM} = \sqrt{\frac{2\text{MgH}}{(M + \frac{I_{CM}}{R^2})}}$ (46)

*Note: If there is a moving system of objects tied by a rope that pass over a pulley without sliding on it, one must take into account the kinetic energy of rotating pulley (**$1/2 * I_{CM\_pulley} * \omega^2$**) during the motion of system.*

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[i] The problem of friction in a rolling motion is more complicated. In more advanced models, one considers that the rolling friction is a mixture of static and kinetic friction.