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LECTURE_13

1] TORQUE VECTOR

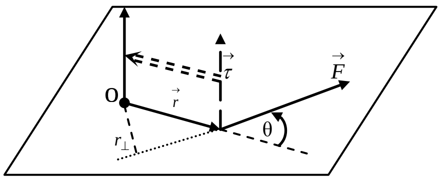

Figure 1. Torque vector

-The torque is a **vector** that *measures* ***rotational action*** due to a force with respect to ***a pivot point***.

$$\vec{\tau} = \vec{r} \, x \, \vec{F} = (rF\sin\theta)\hat{n} \qquad (1)$$

This vector is perpendicular to the plane defined by $\vec{r}$ and $\vec{F}$ (see fig.1).
Its direction (shown by **unit vector** $\hat{n}$) is found by using the right hand rule.
As a rule, one places the **tail** of torque vector at ***the pivot point O.*** The axis that passes by O-point and is aligned to the spatial direction of torque vector represents the axis of the *possible* rotation due to this torque action.

The ***strength of rotational action*** due to a given force $\vec{F}$, i.e. the **magnitude** " $\boldsymbol{\tau}$ " of $\vec{\tau}$ -vector, depends on $\vec{r}$ - vector and the angle **θ .** One may calculate the magnitude of a torque by expression

$$\boldsymbol{\tau} = \boldsymbol{r_\perp} * \boldsymbol{F} \qquad (2)$$

where "$\boldsymbol{r_\perp = r sin\theta}$" is known as the ***arm of force versus O-point.***
Also, one may find the components of ***torque vector*** in any frame by use of matrix rule for cross product (Lect. 4).

2] ROTATION of RIGID BODY at *CONSTANT ANGULAR ACCELERATION* (NEWTON'S SECOND LAW FOR PURE ROTATION)

Assume that a **torque** $\vec{\tau}_{Net}$ rotates CCW a disc at a ***constant angular acceleration*** $\vec{\boldsymbol{\alpha}}$ around a fixed axe. Each particle of disc moves on a circle perpendicular to the axel(fig.2). Let the vector $\vec{F}_i$ be the **net *force*** exerted on $\boldsymbol{i^{th}}$ particle with mass $\boldsymbol{m_i}$. As this particle rotates at angular acceleration of disk $\vec{\boldsymbol{\alpha}}$, besides the **radial** (*centripetal*) **acceleration** $\vec{a}_{i-r}$, it has a **tangential acceleration** $\vec{a}_{i-t}$, too. This means that

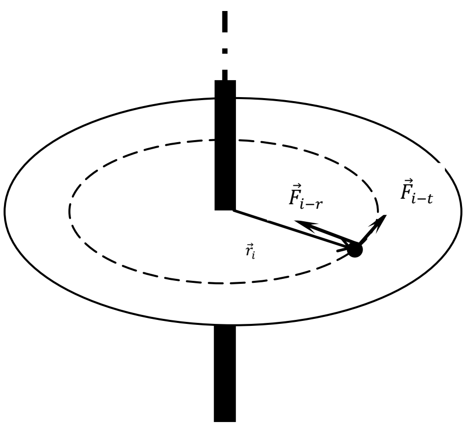

Figure 2 CCW rotation of disk

$$\vec{F}_i = m_i\vec{a}_i = m_i(\vec{a}_{i-r} + \vec{a}_{i-t}) = \vec{F}_{i-r} + \vec{F}_{i-t} \qquad (3)$$

So, the net torque (*versus pivot point on axel*) acting on the i[th] particle is

$$\vec{\tau}_i = \vec{r}_i x \vec{F}_i = \vec{r}_i x \vec{F}_{i-r} + \vec{r}_i x \vec{F}_{i-t} = 0 + \vec{r}_i x \vec{F}_{i-t}$$

$\vec{r}_i x \vec{F}_{i-r} = 0$ *because the angle* 180^o *between them makes* $sin180^0 = 0$

Its magnitude is

$$\tau_i = r_i F_{i-t} sin90^0 = r_i F_{i-t} = r_i m_i a_{i-t} = r_i m_i(\alpha r_i) = m_i r_i^2 \alpha \qquad (4)$$

All the vectors $\vec{\tau}_i$ are aligned along the same direction "up" along axel. So, their sum, that is the ***net external torque*** acting on the whole disc, is aligned along the same direction and its magnitude is equal to the sum of all torques acting on each of its constituent particles m_i. One can find the ***magnitude*** of net torque on disk as

$$\tau_{Net} = \sum_i \tau_i = \sum_i m_i r_i^2 \alpha = \alpha \sum_i m_i r_i^2 = I\alpha \qquad \text{and get} \qquad \tau_{Net} = I\alpha \qquad (5)$$

As the vectors $\vec{\boldsymbol{\alpha}}$ and $\vec{\tau}_{Net}$ are aligned on the same (**axel)** direction, it comes out that

$$\vec{\tau}_{Net} = I\,\vec{\boldsymbol{\alpha}} \qquad (6)$$

The relation (6) describes the rotation of a rigid body at **constant angular acceleration** $\vec{\boldsymbol{\alpha}}$ around a fixed axel under the action of **net external torque** $\vec{\tau}_{Net}$; $\boldsymbol{I}$ is the *inertia moment* of the rigid body about the rotation axel.
As $\boldsymbol{I}$, $\vec{\boldsymbol{\alpha}}$**,** $\vec{\tau}_{Net}$ are parameters used to measure the *inertia*, the *acceleration* and the *net action*, this equation is known as the ***second law of Newton for rotation***.

3] WORK OF NET TORQUE APPLIED ON A RIGID BODY IN ROTATON

If the rotation of a rigid body is *speeding up* due to net torque $\vec{\tau}_{Net}$, the angular speed of body $\boldsymbol{\omega}$ and its kinetic energy $\boldsymbol{K= I\omega^2/2}$ will increase with time. If its center of mass(CM) is on axis of rotation it does not move. So, its gravitational potential energy U_{CM} and translational kinetic energy K_{CM} remain constant. Then, the principle of mechanical energy conservation gives: $W_{ext} = \Delta E_{mech} = (\Delta U_{CM} + \Delta K_{CM}) + \Delta K_{rot} = (0+0) + \Delta K_{rot} = \Delta K_{rot}$.
So, it comes out that the net *external work on the body* goes to change its angular velocity, say, from ω_1 to ω_2, i.e.

$$W_{ext} = \frac{1}{2}I(\omega_2^2 - \omega_1^2) \quad (7)$$

Next, by using the relation $\omega_2^2 - \omega_1^2 = 2\alpha\Delta\theta$ one gets $W_{ext} = \frac{1}{2}I * 2\alpha\Delta\theta = I\alpha\Delta\theta = \tau_{Net} * \Delta\theta$

So, the net work done on an object in <u>*accelerated simple rotation*</u> when it is rotated by Δθ[rad] is

$$W_{\tau-Net} = \tau_{Net} * \Delta\theta \quad (8)$$

Note that in *uniform rotations* (ω is constant, i.e. α = 0), *there is zero net torque* and $W_{ext} = 0$.

4] STATIC EQUILIBRIUM

-Statics deals with ***forces***, ***torques*** and <u>***objects at rest***</u>.
One says that a body is at <u>***translational equilibrium***</u> if the ***acceleration of its CM is zero*** ($\vec{a}_{CM} = 0$).
One says that a body is at <u>***rotational equilibrium***</u> if its ***angular acceleration is zero*** ($\vec{\alpha} = 0$).

Note that in both these two types of equilibrium the body may be <u>*moving, but only in an uniform way*</u> ; i.e. at a ***constant velocity*** or at a ***constant angular velocity***.

A body is in <u>***static equilibrium***</u> (or ***at rest***) if its <u>***translational & angular velocities are both zero***</u>**.**

- A body <u>***at rest***</u> ***(υ=0)*** is for sure in ***translational equilibrium*** as its **CM** has zero acceleration ($\vec{a}_{CM} = 0$).
This means $\vec{F}_{EXT\_net} = M\vec{a}_{CM} = 0$ So, a ***condition for static equilibrium*** is $\vec{F}_{EXT\_net} = \sum \vec{F}_{EXT} = 0$ (9)

By projecting the vector condition (9) on axes Ox, Oy, Oz one can derive ***three algebraic conditions*** from it.

$$\sum F_{EXT-x} = 0 \qquad \sum F_{EXT-y} = 0 \qquad \sum F_{EXT-z} = 0 \quad (10)$$

-A body <u>***at rest***</u> ***(ω=0)*** is for sure *in* ***rotational equilibrium***. So, its ***angular acceleration*** is zero ($\vec{\alpha} = 0$).
From the second law of Newton for rotation one gets $\vec{\tau}_{Net} = I\,\vec{\alpha} = \mathbf{0}$ and this is true <u>***for any pivot point***</u>.
One rewrites this requirement as $\vec{\tau}_{Ext-Net} = \sum_i \vec{\tau}_{i-Ext} = 0$ (11)

If the considered body ***may rotate*** about a given axis, one selects this axe as Oz axis, and the vector condition (11) reduces to ***one algebraic condition*** $\tau_{Ext-Net\_z} = \sum_i \tau_{i-Ext} = 0$ (12)

If the body ***may rotate*** about several axes, one may get several scalar relation similar to (12).

Note: In problems that deal with torques it is *very important to fix a positive direction for rotations and axes and employ all time the same sign convention. In general, one takes* **counter clockwise** *CCW as* **positive** *sense of rotations*.

5] THE CENTER OF GRAVITY

-The ***weight*** is a ***force*** that shows up in all static's problems. It appears in the ***conditions for translational equilibrium*** and its torque appears in the ***condition for rotational equilibrium***. The weight of an object is the ***sum of all gravitation forces*** exerted over any of its constituting particles. These forces are *distributed* over the body extension but " by convention" *one places their sum at the __center of gravity__ of object.* ***Definition***: ***The center of gravity (CG) of an object is the point about which the __net gravitational force__*** ($\vec{F}_{Net-Gravity}$) ***exerted on the object has a __zero torque__.***

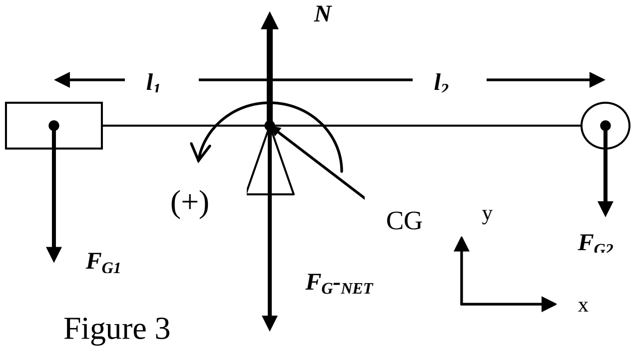

Figure 3

- The ***position*** of CG is critical for the ***rotational equilibrium***. How to find its location? Let´s get **__a rule__** by referring to the "object" in fig.3 (*two "particles 1-2" and a mass less rod that ties them*) ***in rotational equilibrium versus the pivot shown***. This requests a **zero net torque $\vec{\tau}_{Net} = 0$ versus pivot point**. There are three forces applied on this "object" $\vec{F}_{G1}, \vec{F}_{G2}, \vec{N}$. So, rotational equilibrium condition is written:

$$\vec{\tau}_{Net} = \vec{\tau}_N + \vec{\tau}_1 + \vec{\tau}_2 = 0x\vec{N} + \vec{l}_1 x\vec{F}_{G1} + \vec{l}_2 x\vec{F}_{G2} = 0 \qquad (13)$$

Also, when applying the net "object weight " $\vec{F}_{G-Net}$ instead of $\vec{F}_{G1}, \vec{F}_{G1}$, one gets $\vec{\tau}_{Net} = \vec{\tau}_N + \vec{\tau}_{FG-Net} = 0 + \vec{\tau}_{FG-Net} = 0$ and consequently $\vec{\tau}_{FG-Net} = 0$ which places center of gravity "CG" at **pivot point**.

The cross product vectors in relation (13) are perpendicular to the page. One selects the positive direction of rotations as shown, i.e. Oz axis vs. observer and projects the eq.(13) on Oz axe $\quad l_1F_{G1} - l_2F_{G2} = 0 \qquad (14)$

As $\quad l_1 = x_{CG} - x_1 \quad$ and $\quad l_2 = x_2 - x_{CG} \qquad (15)$

one gets $\quad (x_{CG} - x_1)F_{G1} - (x_2 - x_{CG})F_{G2} = 0 \quad$ and $\quad x_{CG}(F_{G1} + F_{G2}) = x_1F_{G1} + x_2F_{G2}$

Finely
$$x_{CG} = \frac{x_1F_{G1} + x_2F_{G2}}{F_{G1} + F_{G2}} \qquad (16)$$

If the object consists of more "particles" the expression (16) becomes $\quad x_{CG} = \frac{\sum_i x_iF_{Gi}}{\sum_i F_{Gi}} = \frac{\sum_i x_iF_{Gi}}{F_{G-Net}} \qquad (17)$

-Note that the expressions for **CM position** are similar to (16) but they refer to "**masses**" while these for **CG position** refer to "**weights**" of object particles. If the object has "*normal size*" such that the *gravitational field* $\vec{g}$ is the same i.e. *constant* for all particles of the object (*all common objects*) one can rewrite (16) as

$$x_{CG} = \frac{\sum_i x_iF_{Gi}}{\sum_i F_{Gi}} = \frac{\sum_i x_i(m_i * g)}{\sum_i m_i * g} = \frac{g\sum_i x_im_i}{g\sum_i m_i} = \frac{\sum_i x_im_i}{\sum_i m_i} = \frac{\sum_i x_im_i}{M} = x_{CM} \qquad (18)$$

This derivation shows that ***__CG and CM are at the same position when the gravitational field__*** $\vec{g}$ ***__is constant over all the object volume__***. Note that ***while CM position of a rigid body is at a fixed location versus object*** and ***does not depend on*** $\vec{g}$, ***CG position depends on*** $\vec{g}$ vector in different positions inside the body volume.

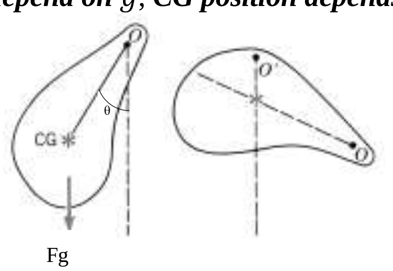

Figure 4

-To **find the CG** of a plane object, one applies the following rule :
"*when __an object is at equilibrium the torque due to its weight must be zero__*".
So, one hangs the body at a *corner point* (O) and passes a ***vertical line through*** the rotation point when body is at rest. ***The CG must be on this line so that sinθ=0 and torque by weight be zero*** (*see fig.4*). Next, one does the same for another point (O') on the corner. The crossing point of two lines gives the location of CG. In the case of non-planar bodies, one needs three points to find the location of CG by use of this technique.
Note: In practice, for common size objects, this is the position of CM, too.