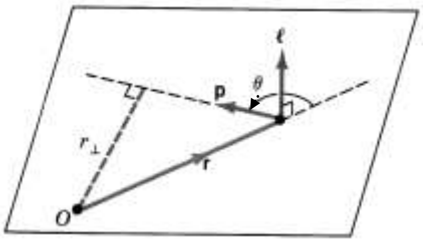
LECTURE_14

1] THE VECTOR OF ANGULAR MOMENTUM



-The ***angular momentum*** is another ***vector parameter*** widely used for the study of rotations. One defines it by a cross product, too. Consider a particle with mass ***m*** at position $\vec{r}$ (*versus origin O* of a *reference frame*) and moving at ***linear momentum*** $\vec{p} = m\vec{\upsilon}$ (*versus the same frame*). One defines its ***angular momentum*** <u>***versus point O***</u> as

Figure 1

$$\vec{l} = \vec{r} \times \vec{p} = (rp\sin\theta)\hat{n} = (r_\perp p)\hat{n}\,[\textbf{kgm}^2/\textbf{s}] \qquad (1)$$

where $\hat{n}$ is a unit vector perpendicular to plane of $\vec{r}$ and $\vec{p}$ vectors.

The quantity " $r\sin\theta \equiv r_\perp$ *" is* ***the*** <u>***arm of linear momentum versus O-point***</u>***.*** *It is the* ***shortest distance*** (i.e. on the perpendicular) *from the* ***reference point*** (O in figure) *to the* ***direction*** *of* ***linear momentum.***

- Let's consider the angular momentum when the particle moves *along a straight line* and on a *circle*.

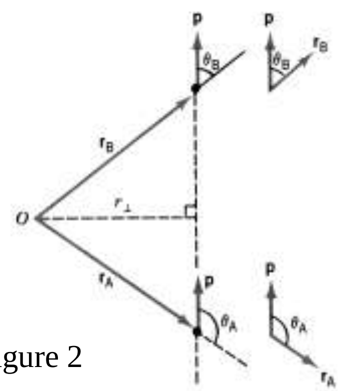


Figure 2

a] Consider a particle moving along a ***straight line*** at ***constant velocity*** (fig.2). The vectors of *linear momentum* at points A and B are equal $\vec{p}_A = \vec{p}_B \equiv \vec{p}$ but $\vec{r}_A \neq \vec{r}_B$. By applying the right hand rule we find out that the vectors $\vec{l}_A = \vec{r}_A \times \vec{p}_A$ and $\vec{l}_B = \vec{r}_B \times \vec{p}_B$ are perpendicular to the page and directed versus observer. Their magnitudes are equal; $l_A = p_A(r_A \sin\theta_A) = p_A r_\perp = p r_\perp$ (see fig.2) and $l_B = p_B(r_B\sin\theta_B) = p_B r_\perp = p\ r_\perp = l_A$. So, when ***a particle moves at constant velocity, its angular momentum "versus reference point O" is*** <u>***constant all time.***</u>

If the particle is moving **along** $\vec{r}$ direction, its angular momentum is **zero all time**.

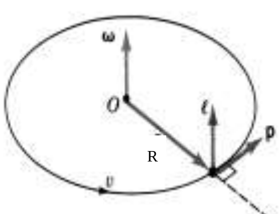


Figure 3

b] Let's consider now a particle moving at ***constant speed*** on a **circular path**.
If the *origin of the reference frame* is *at* the *circle center* (fig. 3),the angle between $\vec{r}$ *and* $\vec{v}$, $\theta = 90^0$ all time. The *angular momentum* vector is directed *perpendicular to the plan of the circle* and (*since* $\sin 90^0 = 1$) its ***magnitude*** is

$$l = Rp\sin 90^0 = Rp = Rmv \qquad (2)$$

As $v = R\omega$ one gets $\qquad l = mR^2\omega = I_z\omega \qquad (3)$

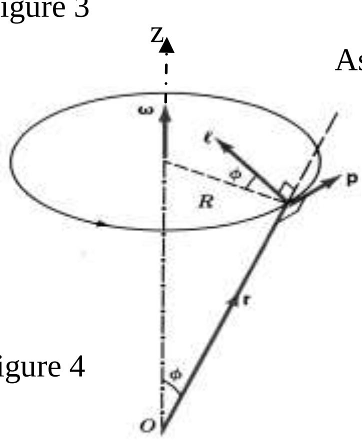


Figure 4

In the case of a system of particles in space, the origin of frame cannot be at the circle center for all of them. So, in general, the angular momentum vector $\vec{l} = \vec{r} \times \vec{p}$ is not perpendicular to the plane of the particle motion circle (fig.4); and ***it is not parallel to vector*** $\vec{\boldsymbol{\omega}}$. But, the **component** vector of $\vec{l}$ **along Oz** axis $\vec{l}_z$ is always **paralle**l to $\vec{\omega}$ vector **no matter what is the direction** of $\vec{l}$. In the following , one looks for a relation between vectors "$\vec{\omega}$" and "$\vec{l}_z$".

Remember: In rotation problems, one selects Oz axis such that $\vec{\boldsymbol{\omega}}$ ***lies on it.***

Since, $\vec{p} \perp \vec{r}$ (*because $\vec{p}$ is tangent to the circular base of cone and $\vec{r}$ is $\perp$ to circle tangent, see fig.4*), the angle between them is 90^o and the ***magnitude of angular momentum*** is $l = rpsin90^o = rp = rm\upsilon$ (4)

Then, since $sin\,\varphi = R/r$ (see fig. 4) $$l_z = l\sin\phi = l\frac{R}{r} = rm\upsilon\frac{R}{r} = m\upsilon R = m(\omega R)R = mR^2\omega$$

So, one gets $$l_z = mR^2\omega = I_z\omega \qquad (5)$$

Note that expression (5) shows that magnitude of ***z-component*** of angular momentum $\vec{l}_z$, for a particle in rotation around an axis Oz, depends on its inertia moment versus this axis and its angular velocity $\boldsymbol{\omega}$.

- The **angular momentum** for a **system of particles** is defined as $\vec{L} = \sum_i \vec{l}_i$ (6) , so $L_z = \sum_i l_{iz}$ (7)

is its **z-component.** If all particles rotate about the **same axis at same** $\vec{\omega}$, by applying the relation (5) for the i-th particle, one gets $l_{iz} = m_i R_i^2 \omega$ and, by using (7) one finds out that $L_z = \omega \sum_i m_i R_i^2$ (8)

R_i is the distance of i^{th} particle from axis of rotation.

As the quantity $\sum_i m_i R_i^2 = I_z$ gives the inertia moment of the system of particles versus Oz axis, the expression (8) takes the form $$L_z = I_z\omega \qquad (9)$$

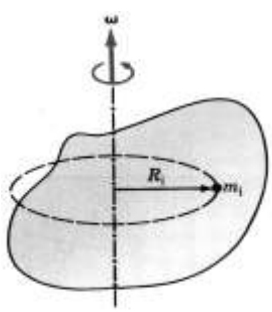


Figure 5

-In the case of a **rigid body rotating around a fixed axis** (fig.5), R_i^2 values do not change in time. So, $\sum_i m_i R_i^2 = I_z$ is a constant parameter. If a rigid body rotates at angular velocity $\vec{\omega}$, one places the axe Oz along direction of $\vec{\omega}$ and the angular momentum $\vec{L}$ is aligned with Oz. In this case the expression (9) can be written as

$$\vec{L} = I_z\vec{\omega} \qquad (10)$$

- If the system of particles does ***not*** form a ***rigid body*** (ex. a *liquid inside a rotating container*) the distances R_i of particles change in time and the inertia moment $\boldsymbol{I_z}$ of the object is a function of time. Meanwhile, *if all the particles* **rotate at same angular velocity**, one may still apply the formulas 6-7-8-9 but keeping in mind that, in this case, ***Iz*** and ***Lz*** are both parameters that can change with time.

2] ROTATION DYNAMICS AND THE CONSERVATION OF ANGULAR MOMENTUM

-The angular momentum of a ***single particle*** versus a point O(see 1) is defined as $\vec{l} = \vec{r} \times \vec{p}$ (11)

The derivative of (11) gives $$\frac{d\vec{l}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{\upsilon} \times m\vec{\upsilon} + \vec{r} \times \vec{F} \quad \text{and} \quad \frac{d\vec{l}}{dt} = \vec{r} \times \vec{F} = \vec{\tau} \qquad (12)$$

(because $\vec{\upsilon} \times m\vec{\upsilon} = 0$ and $\frac{d\vec{p}}{dt} = \vec{F}$ in modern way of writing the second law of Newton).

-By applying (12) for the ***i**^th^ **particle*** of a set of particles, one gets $\vec{\tau}_i = \vec{r}_i \times \vec{F}_i = \dfrac{d\vec{l}_i}{dt}$ (13)

Then, the ***net external torque*** exerted on the ***set of particles*** by the external action (that makes them rotate)

is the sum of all torques exerted on those particles $\vec{\tau}_{NET} = \sum_i \vec{\tau}_i = \sum_i \dfrac{d\vec{l}_i}{dt} = \dfrac{d}{dt}\sum_i \vec{l}_i = \dfrac{d\vec{L}}{dt}$

So, one get

$$\vec{\tau}_{NET} = \frac{d\vec{L}}{dt} \quad (14)$$

$\vec{L} = \sum_i \vec{l}_i$ is the ***net angular momentum of the set of particles;*** $\vec{\tau}_{NET}$ is the ***net external torque*** on them.

If the set of particles constitutes *a rigid body*, then inertia moment I is constant, $\vec{L} = I\vec{\omega}$ and the relation (14) transforms into $\vec{\tau}_{NET} = \frac{d\vec{L}}{dt} = \frac{d}{dt}(I\vec{\omega}) = I\frac{d\vec{\omega}}{dt} = I\vec{\alpha}$ *which is the second law for rotation.* Note that the expression (14) has a larger range of applications than the 2^{nd} law for rotations because it is valid even in situations when inertia moment of *system may* change *during rotations.*

- If the ***net torque exerted on the set of particles*** is zero, from (14) $\dfrac{d\vec{L}}{dt} = 0 \Rightarrow\Rightarrow \vec{L} = const$ (15)

The expression (15) is the mathematical expression of the *principle of angular momentum conservation*.

If the net external torque** (versus a reference point or axis) **exerted on a system of particles is zero, then the vector of angular momentum of the system** (versus this point or axis) **remains constant in time.

In majority of scenarios, all particles of set (*or a part of them like in ex.1*) would be rotating (*at least for a while*) at *same angular velocity* around *a given axis*. So, one places Oz along this axis and by projecting the relation (14) on Oz axis, one gets

$$\tau_{NET-z} = \frac{dL_z}{dt} \quad (16)$$

If the ***net external torque*** $\boldsymbol{\tau_{Net-z} = 0}$, one finds out that $\dfrac{dL_z}{dt} = 0$ i.e. ***Lz = const***.

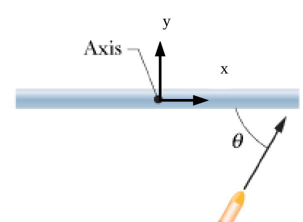
So, for *any two moments of time* $\mathbf{t_2}$ and $\mathbf{t_1}$

$$L_{set-z}(t_2) = L_{set-z}(t_1) \quad (17)$$

Let's consider a set of particles that **rotate** around the same ***fixed axis Oz.*** If external net torque versus this axis is zero, the ***angular momentum of the set*** versus this axe will ***remain constant*** in time. Then, from relation (9) $L_z = I_z * \omega$ comes out that, this ***system of particles*** may change its inertia moment or its angular velocity or both of them but their product will remain constant in time; i.e. if $\boldsymbol{\tau_{Net-z} = 0}$ then $\boldsymbol{L_{sys-z} = c^{te}}$ and

$$I_z(t_2) * \omega(t_2) = I_z(t_1) * \omega(t_1) \quad (18)$$

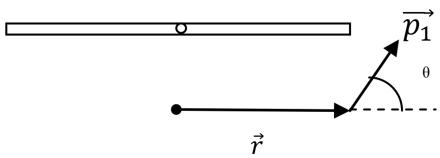
***Exemple 1**. A uniform thin rod of length **l = 0.5 m** and mass M_{rod} = **4kg** can rotate in a horizontal plane about a vertical axis passing through its center. The rod is at rest when a **m_b = 3.0 g** bullet traveling in the rotation plane is fired into one end of the rod. As viewed from above, the bullet's path makes an angle **θ = 60°** with the rod. If the bullet lodges in the rod and the angular velocity of the rod is **5 rad/s** immediately after the collision, what is the bullet's speed just before impact?*



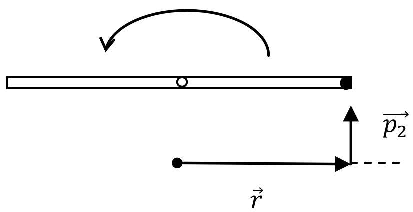
***Step 1.** Objects do not move together before collision **but will rotate together around Oz axis** after collision. So, one must consider separately and get the sum of **angular momentums of the set of two objects before and after collision**.*

Step 2. *One notes that, after the collision, the set rod-bullet will rotate CCW around the axis* **Oz**. *So, one selects Oz axe directed versus observer. Next , one writes (16) for the* **system rod - bullet** $\tau_{NET-z} = \frac{dL_z}{dt}$ *and notes that actions from outside the system produce* $\boldsymbol{\tau_{Net-z} = 0}$ because:
a) the weight of rod and normal force on its CM (by pivot) are applied at zero distance from Oz (i.e. zero torque);
b) the weight of bullet (directed along - Oz) has a torque vector in plane Oxy (i.e. zero component on Oz direction).

Step 3. *As* $\boldsymbol{\tau_{Net-z} = 0}$ *one may apply relation (18) for two moments; one* <u>*before*</u> *and the other one* <u>*after*</u> *collision.*



At the moment t_1 *(just before collision) the z- component of total linear momentum of set is* $L_{z-1} = L_{rod-1} + L_{b-1}$
$L_{rod-1} = 0$ *because the rod is* <u>*at rest*</u> *;*
$L_{b-1} = rp_1\sin\theta = (l/2)(m_b\upsilon_1)\sin60^0$; So $L_{z-1} = 0.5lm_b\upsilon_1\sin60^0$ (19)



At the moment t_2 *(just after collision) the z- component of total linear momentum of set is* $L_{z-2} = L_{rod\&b-2} = I_{rod\&b-2} * \omega$

$I_{rod\&b-2} = I_{rod} + I_b = M_{rod}\, l^2/12 + m_b(l/2)^2 = l^2(M_{rod}/12 + m_b/4)$

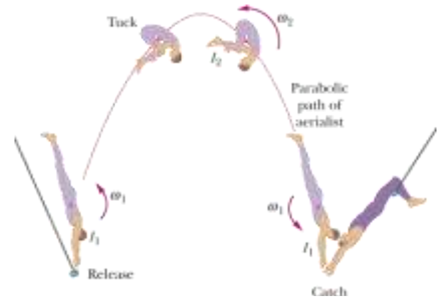
Then, $L_{z-2} = l^2(M_{rod}/12 + m_b/4)*\omega$ (20)

Step 4. *As the angular momentum of the set is conserved, one can substitute (19) and (20) at relation (17) i.e.* $L_{sys-z}(t_2) = L_{sys-z}(t_1)$ *and get the relation*

$$0.5*l*m_b\upsilon_1*\sin60^0 = l^2(M_{rod}/12 + m_b/4]*\omega$$ *from which it comes out that*

$$\upsilon_1 = l(M_{rod}/12 + m_b/4]*\omega \;/\; 0.5*m_b*\sin60^0 = 0.5(4/12 + 0.003/4)*5 \;/\; 0.5*0.003*\sin60^0 = 642.94 m/s$$

<u>***Example 2***</u>. *During a jump, an aerialist makes* ***four revolutions*** *for* ***1.87s*** *before attaching his hands to team partner. For the* ***first*** *and* ***last quarter-revolution***, *he is in the extended orientation as shown in figure, with a rotational inertia moment* $\boldsymbol{I_1 = 19.9 kgm^2}$ *around his center of mass (CM shown by a red dot in fig). During the rest of the flight he is in a tight tuck, with* $\boldsymbol{I_2 = 3.93 kgm^2}$.
What is his angular speed $\boldsymbol{\omega_2}$ *around his center of mass during the tuck?*



Step 1. *One observes that the aerialist is participating in a composed motion;*
a) a projectile motion of his CM versus the earth frame; His CM will follow a parabola(<u>*the problem is not asking any question about this motion*</u>*).*
b) a rotational motion versus an Oz axe passing by his center of mass(<u>*the question is related with this motion*</u>*).*

Step 2.*One might note that the location of his CM versus the body does change during the rotation and the problem is giving the values of his moment of inertia versus CM. So, one consider the model "****a set of particles that changes it inertia moment versus CM during rotations around axis Oz out of plane and passing by its CM*** ". As the only applied force**, *aerialist gravity,*** has ***zero torque versus CM and all particles of the "object" rotate with the same angular velocity,*** *one may apply (17) for three positions and get* $I_1\omega_1 = I_2\omega_2 = I_3\omega_3$ (21)

Step 3. *From relation (21) one gets* $\omega_1 = (I_2/I_1)*\omega_2$ (22) *and* $\omega_3 = \omega_1$ (23) *because* $I_3 = I_1$

Step 4. *Assuming a* **constant angular velocity** *during each part of rotations, one gets*
$\Delta t_1 = \Delta\theta_1/\omega_1 = (2\pi/4)/\omega_1 = 0.5\pi/(I_2/I_1)*\omega_2$; $\Delta t_3 = \Delta\theta_3/\omega_3 = \Delta\theta_1/\omega_1$ .. $\Delta t_3 = (2\pi/4)/\omega_1 = 0.5\pi/(I_2/I_1)*\omega_2$
and $\Delta t_2 = \Delta\theta_2/\omega_2$ .... $\Delta t_2 = (4*2\pi - 2*0.5\pi)/\omega_2 = 7\pi/\omega_2$. *Next, as* $t_{tot} = \Delta t_1 + \Delta t_2 + \Delta t_3$ (24)
$t_{tot} = 0.5\pi/(I_2/I_1)*\omega_2 + 7\pi/\omega_2 + 0.5\pi/(I_2/I_1)*\omega_2 = \pi/(I_2/I_1)*\omega_2 + 7\pi/\omega_2 = (\pi/\omega_2)[(I_1/I_2) + 7]$
Finally $\omega_2 = (\pi/t_{tot})[(I_1/I_2) + 7] = (3.14/1.87s)(19.9/3.93 + 7) = 20.26$ r/s