LECTURE 14

1] NEWTON'S LAW OF GRAVITY

- Isaac Newton has formulated ***gravity law*** for two ***point particles*** with masses m_1 and m_2 at a distance ***r*** between them. Here are the four steps that bring to universal law of gravitation.

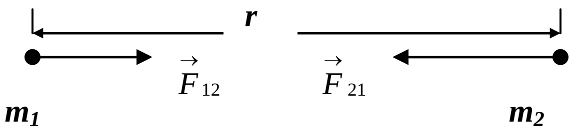

Figure 1

a) Based on experimental information one postulates that ***gravitation produces*** an ***attractive force***.

b) Based on the second law of Newton
$F_{12} = m_1 a_1 \_so\_ F_{12} \approx m_1$ similarly $F_{21} \approx m_2$

c) Based on the third law of Newton $F_{12} = F_{12} \equiv F$. Then, it comes out that $F \approx m_1 m_2$ (1)

d) Based on experimental results, Newton guessed that gravitation force decreases with distance as ***1/r²***.

So, it comes out that $F \approx \frac{m_1 m_2}{r^2}$ and by labelling as "G" proportionality constant $F = G\frac{m_1 m_2}{r^2}$ (2)

The experiment shows that the universal constant of gravitation has the value G = $6.67 \cdot 10^{-11}$ Nm^2/kg^2

- The gravitational force is a ***vector directed versus the source that exerts this force***. So, its vector form is

$$\vec{F}_{12} = -G\frac{m_1 m_2}{r^2}\frac{\vec{r}_{21}}{r_{21}} \quad (3)$$

$\frac{\vec{r}_{21}}{r_{21}}$ is the unit vector with tail at mass ***m₂*** – the **source** of $\vec{F}_{12}$.

$$\vec{F}_{21} = -G\frac{m_1 m_2}{r^2}\frac{\vec{r}_{12}}{r_{12}}$$

$\frac{\vec{r}_{12}}{r_{12}}$ is the unit vector with tail at mass ***m₁***– the **source** of $\vec{F}_{21}$

- To apply the gravitational law for two bodies close to each other one must use integration techniques and the difficulty of calculi depends on the form of the two bodies. But, if the bodied are far enough to each other[1], one may model them as point particles and apply the law in its original form. In particular, with some approximations, we are can model the interaction of earth with an object near to its surface as if the whole mass of the earth is concentrated at its center and the "point object" is at distance ***$r = R_{earth}$***.

-Many experiments have shown that; when several particles interact gravitationally between them the ***principle of linear superposition applies***. So, inside a system of particles ***$m_1, m_2, m_3, \ldots m_n$*** , the force exerted on mass ***m₁*** is

$$\vec{F}_1 = \sum_{i=2}^{n} \vec{F}_{1i} \quad (4)$$

2] THE GRAVITATIONAL MASS AND THE INERTIAL MASS

-When expressing the ***second law*** of Newton we use the ***inertial mass*** $\vec{F}_{NET} = m_{in}\vec{a}$ (5)

When formulating the ***gravitation law***, Newton was ***not sure*** that the mass of particles in this law is the same as their ***inertial mass***. To clarify this issue, one may start by assuming that the gravitational mass ***m_{gr}*** be different from ***m_{in}*** and consider a body in free fall close to earth surface. The earth will exert on it the gravitational force with magnitude

$$F_{gr} = G\frac{m_{gr} M_{Earth}}{R_{Earth}^2} \quad (6)$$

Here we assume that the body is close to the surface so that its distance form earth center is $\cong R_{Earth}$.

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[1] Their dimensions are <<< than the distance between them.

As this is the net force exerted on the body, one apply the second law of Newton and get

$$F_{NET} = m_{in}a = F_{gr} = G\frac{m_{gr}M_{Earth}}{R_{Earth}^2} \quad \text{So,} \quad m_{in}a = m_{gr}(G\frac{M_{Earth}}{R_{Earth}^2}) = m_{gr}g \quad \text{where } G\frac{M_{Earth}}{R_{Earth}^2} = g \quad (7)$$

Then,

$$a = \frac{m_{gr}}{m_{in}}g \quad (8)$$

The measurements show that the free fall acceleration of bodies near earth surface is equal to a= ***9.8m/s²***. Then the expression (8) tells that one can get a = ***9.8m/s²*** if ***m_{gr}/m_{in} =1*** or ***$m_{gr} = m_{in}$*** and the expression (7) is equal to 9.8m/s^2. If one uses the values of earth mass and earth radius of in (7) it comes out that G has the value 6.67*10^{-11}Nm^2/kg^2 . So, for this G-value, the ***experiments confirm that the gravitational mass is the same as the inertial mass***.

- Let's apply the gravitation law for the force exerted by earth on a mass 1kg close to earth(small h-values).

$$\vec{F} = -G\frac{M_{Earth}}{(R_{Earth}+h)^2}\frac{\vec{R}_{Earth}}{R_{Earth}} \equiv \vec{g} \quad (9)$$

So, the <u>g-vector is equal to the gravitation force exerted on a mass 1kg</u>. By measuring the gravitation force on 1kg mass at different locations near/on earth one gets a <u>system of g-vectors</u> (fig.2). ***The totality of these vectors forms the gravitational field of the earth.***

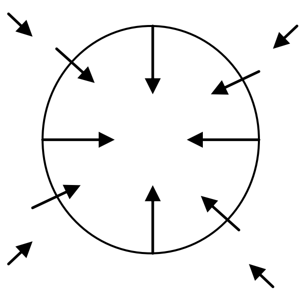

Figure 2

Note that the g-vector magnitude decreases with the increase of distance "***h***" from earth surface but it is always directed versus the center of the earth. The gravitational field of the earth has a spherical symmetry. *In fact, it is not exactly spherical, because the model of earth as a uniform density sphere is not very precise*. Now, the object weight is equal to gravitational force exerted by this field

$$\vec{W} = m\vec{g} \quad (10)$$

So, the weight of the same object is a vector that is different in different points of gravitational field of the earth.

3] KEPLER'S LAWS ON PLANETARY MOTION

- *First law*: <u>*The planets move on elliptic orbits around the sun that is located at one of its focuses*</u>.

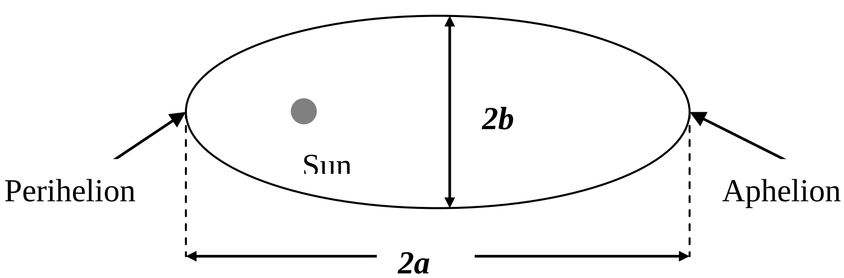

Figure 3

The minor axis is long 2b and the major axis is long 2a. The closest distance to sun is called ***perihelion*** and the biggest distance to sun is called ***aphelion*** (fig.3).

- *Second law*: <u>*The line sun- planet sweeps out equal areas for equal interval of times*</u>.

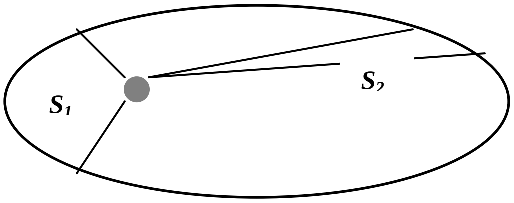

Figure 4

- *Third law*: <u>*The square of the period of planet motion is proportional to the cube of average distance from the sun*</u>. Calculations show that the <u>**average of distance**</u> sun-planet is equal to half of major axis ***a***. Then, satellite motion (see Lecture_8.5) tells that

$$T^2 = \kappa a^3 \quad (11)$$

where $\kappa_{sun} = \frac{4\pi^2}{GM_{sun}}$

Note : Kepler's laws are valid for elliptical paths of any planet around a central body; for example the moon moving around the earth but in this case $\kappa_{Earth} = \dfrac{4\pi^2}{GM_{Earth}}$.

THE ENERGY OF PLANETS

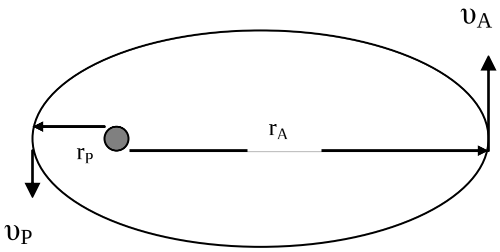

Figure 5

-As the mass of other planets is much smaller than the mass of sun we neglect their action on the motion of the studied planet. We consider that the *system sun-planet is a conservative system*, i.e. the forces originated from outside it are zero. In these circumstances:
a) The torque of exterior force is zero and we can apply the ***principle of conservation of angular momentum.***
b) The work done by Net exterior force is zero and we can apply the ***principle of energy conservation for the system sun-planet***

- The principle of angular momentum conservation tells $\vec{L}_A = \vec{L}_P$ (12) or $\vec{r_A} x \vec{p}_A = \vec{r_P} x \vec{p}_P$ (13)
The equality of magnitudes brings to condition $r_A p_A \sin 90^0 = r_P p_P \sin 90^0 \Rightarrow r_A m_{planet} \upsilon_A = r_P m_{planet} \upsilon_P$

So, one get $$r_A v_A = r_P v_P \quad (14)$$

-The principle of energy conservation tells that that $$\boldsymbol{E_A = E_P} \quad (15)$$

As the mechanical energy is E = K + U where $K = \dfrac{m_{pl}\upsilon_{pl}^2}{2}$ (16) and $U = -G\dfrac{m_{pl}M_{Sun}}{r_{pl-Sun}}$ (17)

one get $$\frac{m_{pl}\upsilon_A^2}{2} - G\frac{m_{pl}M_{Sun}}{r_A} = \frac{m_{pl}\upsilon_P^2}{2} - G\frac{m_{pl}M_{Sun}}{r_P} \quad (17)$$

After cancelling m_{pl} : $\dfrac{\upsilon_A^2}{2} - G\dfrac{M_{Sun}}{r_A} = \dfrac{\upsilon_P^2}{2} - G\dfrac{M_{Sun}}{r_P}$ and finally $2GM_{Sun}(\dfrac{1}{r_P} - \dfrac{1}{r_A}) = \upsilon_P^2 - \upsilon_A^2$ (18)

By using the equation (14) and the fact that $r_A + r_P = 2a$ (19), after some calculations we find that

$$\upsilon_A^2 = \frac{GM}{a}\frac{r_P}{r_A} \quad (20) \qquad \text{and} \qquad \upsilon_P^2 = \frac{GM}{a}\frac{r_A}{r_P} \quad (21)$$

Finally, by substituting one of this expressions at the expression of total energy (at perihelion or aphelion) we get $$E = -G\frac{m_{pl}M_{Sun}}{2a} \quad (22)$$

4] THE BOUND AND UNBOUND TRAJECTORIES

-Let see what happens to an object with mass "*m*" thrown vertically up with large *initial velocity* "$\boldsymbol{\upsilon}$".
The only force exerted on it is the gravitational force of the earth with magnitude

$$F_G = G\frac{m\, M_{Earth}}{r^2} \quad (23)$$
"*r*" stands for its distance from center of earth

As external net force is zero, the *system earth – object* ***conserves its energy***. For such a system

Note: As the earth is the reference frame, generally one talk for object energy.

$$\boldsymbol{E = U + K} \quad \text{is a constant all time.}$$

- By using the relation between the potential energy function and conservative force ($F = -\frac{dU}{dr}$) one may find out that

$$U(r) = -G\frac{mM_{Earth}}{r} \qquad (24)$$

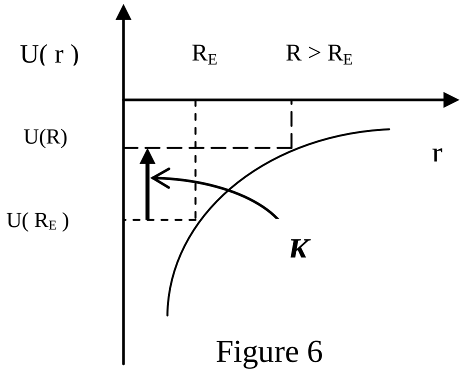

Figure 6

The graph of this function is shown in figure 6. Note that the zero value of potential energy of gravitational interaction is met for big r-values . ***The zero-value of potential energy means that there is no interaction between the system parts, or more precisely the parts are not bound to each other***. When the object is on the earth surface, the distance from CM of the earth (located at origin of axes) is equal to the earth radius R_E and its potential energy is $U(R_E)$. When we throw it vertically with initial velocity ***υ,*** the mechanical energy of the object (i.e. of object earth system) is

$$\boldsymbol{E(R_E) = U(R_E) + K(\upsilon)} \qquad (25)$$

As we know, the ***magnitude of velocity*** will decrease with height and will ***become zero*** at its maximum height ***h***, i.e. at distance $\boldsymbol{R = (R_E + h)}$. When getting at this distance, all its energy is potential energy. So,

$\boldsymbol{E(R) = U(R)}$ (26) *We say that the kinetic energy makes it climbs up on a potential well* (see fig.6).

The principle of energy conservation tells that $\boldsymbol{E(R) = E(R_E)}$ (27)

By using this equation we can find the maximum height as follows; $\boldsymbol{E(R_E+h) = E(R_E)}$

As $\boldsymbol{E(R_E+h) = U(R_E+h)}$ ***and using (25)*** we get $\boldsymbol{U(R_E+h) = E(R_E) = U(R_E)+K(\upsilon)}$ (28)

So, $-G\frac{mM_{Earth}}{R_E + h} = -G\frac{mM_{Earth}}{R_E} + \frac{m\upsilon^2}{2}$ and $GM_{Earth}(\frac{1}{R_E} - \frac{1}{R_E + h}) = \frac{\upsilon^2}{2}$

Then, $GM_{Earth} * \frac{1}{R_E}\left(1 - \frac{1}{1+\frac{h}{R_E}}\right) = \frac{GM_{Earth}}{R_E}\left(\frac{1+\frac{h}{R_E}-1}{1+\frac{h}{R_E}}\right) = \frac{GM_{Earth}}{R_E}\left(\frac{\frac{h}{R_E}}{1+\frac{h}{R_E}}\right) = \frac{GM_{Earth}}{R_E^2}h$ as $\frac{h}{R_E} \ll 1$

So, one get $\frac{GM_{Earth}}{R_E^2}h = \frac{\upsilon^2}{2}$ and as $\frac{GM_{Earth}}{R_e^2} = g$ we find that $h = \frac{\upsilon^2}{2g}$ (29) as known in kinematics.

- If the initial velocity "***υ***" of the body is such that $\boldsymbol{U(R_E) + K(\upsilon) < 0}$ then the object will arrive till to a given distance "$h=h_{max}$" from earth surface but will remain ***all time bound*** within the system earth-object; But, if initial velocity "***υ***" is such that $\boldsymbol{U(R_E) + K(\upsilon) \geq 0}$ ***, then*** the object will go so far that the interaction with earth becomes zero; it becomes an ***unbounded*** object to gravitational field of the earth. The limiting initial velocity necessary to unbind from gravitational field of earth is known as ***escape velocity*** "$\boldsymbol{\upsilon_{esc}}$"**.** This velocity can be found by condition

$$U(R_E) + K(\upsilon_{esc}) = 0 \Rightarrow -G\frac{mM_E}{R_E} + \frac{m\upsilon_{esc}^2}{2} = 0 \Rightarrow\Rightarrow\Rightarrow\Rightarrow\Rightarrow \upsilon_{esc} = \sqrt{\frac{2GM_E}{R_E}} \qquad (29)$$

- Suppose that one must send a rocket out of earth gravitational field; i.e. ***make it an unbound*** object to earth. The first requirement is to give to rocket an initial velocity $\upsilon \geq \upsilon_{esc}$. A simple calculation based on (28) shows that $\boldsymbol{\upsilon_{esc}}$ **=11190 m/s**. This value of initial speed can be realized only by reactive engines.

What form has the trajectory of an unbound rocket? The mathematical calculations show that if

a) $\upsilon = \upsilon_{esc}$ the rocket trajectory will ***not be closed*** and it is a ***parabola***
b) $\upsilon > \upsilon_{esc}$ the rocket trajectory will ***not be closed*** and it is a ***hyperbola***.

If the initial velocity of rocket $\upsilon < \upsilon_{esc}$ the object remains a bound object to the earth gravitation field and its ***trajectory*** will be an ***ellipse*** (***closed orbit***). Note that if the initial velocity is too small $\upsilon \ll \upsilon_{esc}$, this orbit may cross the earth, i.e. the rocket will strike on earth surface (figure 7).

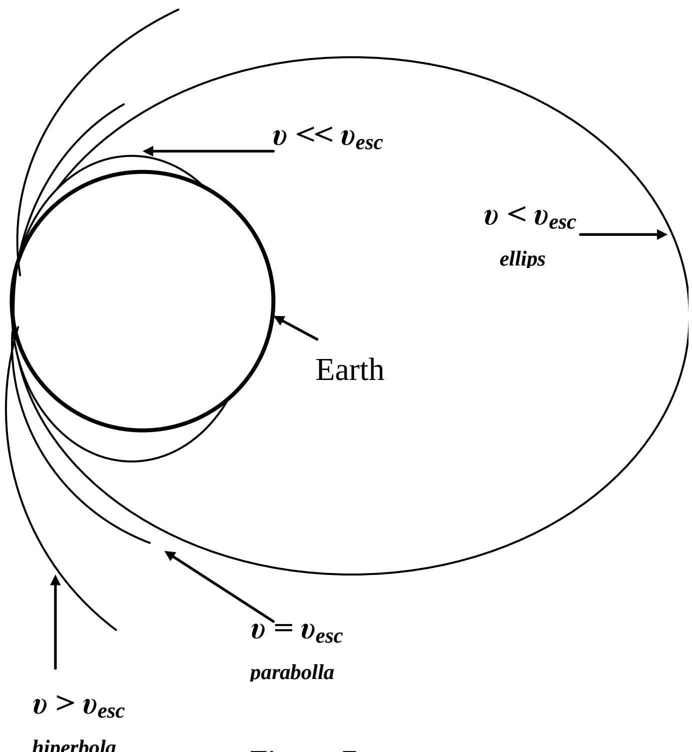

Figure 7

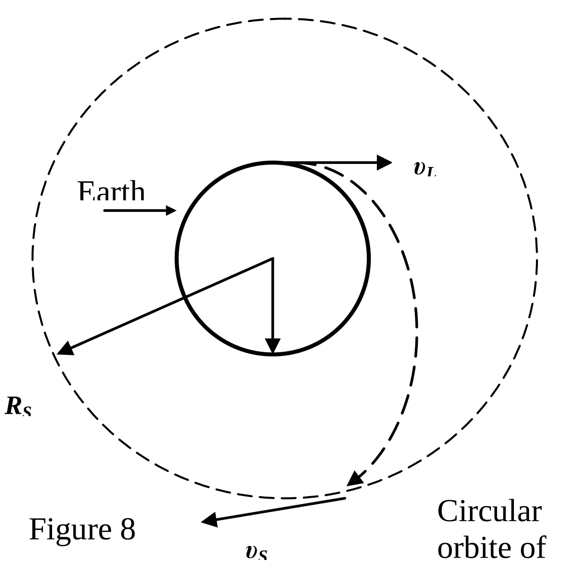

Figure 8

Circular orbite of Satellite.

-Consider an artificial satellite moving uniformly on a circular orbit at distance R_S from CM of the earth (fig.8). Its motion has a centripetal acceleration $a_c = \dfrac{\upsilon_s^2}{R_S}$ and the net force exerted on it is the earth gravitation. As $F = G\dfrac{m_s M_E}{R_s^2}$ and $F = m_s a_c = m_s \dfrac{\upsilon_s^2}{R_S}$ one get $G\dfrac{m_s M_E}{R_s^2} = m_s \dfrac{\upsilon_s^2}{R_S}$

and
$$\upsilon_s^2 = \frac{GM_E}{R_S} \qquad (30)$$

What is the required **launch speed** $\boldsymbol{\upsilon_L}$ for this satellite?
We apply the principle of energy conservation.

$$\boldsymbol{E(R_E) = E(R_S)} \qquad (31)$$

$$\frac{m_s \upsilon_L^{\ 2}}{2} - G\frac{m_S M_E}{R_E} = \frac{m_S \upsilon_s^2}{2} - G\frac{m_s M_E}{R_S} = \frac{m_S}{2}\frac{GM_E}{R_S} - G\frac{m_s M_E}{R_S}$$

So, $$\frac{\upsilon_L^{\ 2}}{2} = G\frac{M_E}{R_E} - G\frac{M_E}{2R_S} = G\frac{M_E}{R_S}\left(\frac{R_S}{R_E} - \frac{1}{2}\right) \qquad (32)$$

$$\upsilon_L^{\ 2} = 2G\frac{M_E}{R_S}\left(\frac{R_S}{R_E} - \frac{1}{2}\right) = 2G\frac{M_E}{R_S}\left(\frac{R_E + h}{R_E} - \frac{1}{2}\right) =$$
$$= 2G\frac{M_E}{R_S}\left(1 + \frac{h}{R_E} - \frac{1}{2}\right) = G\frac{M_E}{R_S}\left(1 + \frac{2h}{R_E}\right) \qquad (33)$$

For *low orbit* satellites (say heights100 - 200Km from earth) $R_S \approx R_E$(**6378km**) $2h/R_E \approx 0$ and from (32) we can find that

$\upsilon_L^2 \cong G\dfrac{M_E}{R_S}$ So, the launch velocity for a stationary circular low orbit with radius R_S is
$$\upsilon_L \approx \sqrt{\frac{GM_E}{R_S}} \qquad (34)$$