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LECTURE 15

1] NEWTON'S LAW OF GRAVITATION

-The ***gravitation law*** is formulated for two ***point particles*** with masses m_1 and m_2 at a distance $\boldsymbol{r}$ between them. Here are the four steps that bring to the Universal Law of Gravitation discovered by Isaac NEWTON.

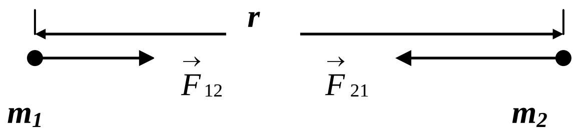

Figure 1

a) Based on experimental observations one postulates that ***gravity*** is a non contact ***attractive force*** *(see fig.1)*.

b) Based on the second law of Newton
$F_{12} = m_1 a_1 \_so\_ F_{12} \approx m_1$ and similarly $F_{21} \approx m_2$

c) Based on the third law of Newton $F_{12} = F_{12} \equiv F$. So, it comes out that $$F \approx m_1 m_2 \qquad (1)$$

d) Based on experimental results, Newton guessed that gravitation force decreases with distance as $\boldsymbol{1/r^2}$.

So, it comes out that $F \approx \frac{m_1 m_2}{r^2}$ and more precisely $$F = G\frac{m_1 m_2}{r^2} \qquad (2)$$

Measurements show that numerical value of Universal Constant of Gravitation is $G = 6.67*10^{-11} Nm^2/kg^2$

- As gravitational force is a ***vector directed versus the source that exerts this force***, its vector form is

$$\vec{F}_{12} = -G\frac{m_1 m_2}{r^2}\frac{\vec{r}_{21}}{r_{21}} \qquad (3)$$

$\frac{\vec{r}_{21}}{r_{21}}$ is the unit vector with tail at mass $\boldsymbol{m_2}$ – the **source** of $\vec{F}_{12}$.

$$\vec{F}_{21} = -G\frac{m_1 m_2}{r^2}\frac{\vec{r}_{12}}{r_{12}}$$

$\frac{\vec{r}_{12}}{r_{12}}$ is the unit vector with tail at mass $\boldsymbol{m_1}$ – the **source** of $\vec{F}_{21}$.

- To apply the gravitational law for two bodies close to each other one must use integration techniques and the difficulty of calculi depends on the shapes of the two bodies. But, if the bodied are far enough to each other[1], one may model them as point particles and apply the law in its original form. In particular, within some approximations, one can model the interaction of earth with an object on its surface as if the whole mass of the earth is concentrated at its center and the objects are at distance $\boldsymbol{r = R_{earth}}$.

-Many experiments have shown that when several objects interact between them by gravitational forces, the ***principle of linear superposition*** applies. So, in the case of a system of particles $\boldsymbol{m_1, m_2, m_3, \ldots m_n}$, the net gravitational force exerted on mass $\boldsymbol{m_1}$ is $$\vec{F}_1 = \sum_{i=2}^{n} \vec{F}_{1i} \qquad (4)$$

2] THE GRAVITATIONAL MASS AND THE INERTIAL MASS

-When writing the ***second law*** of Newton one refers to the ***inertial mass*** $$\vec{F}_{NET} = m_{in}\vec{a} \qquad (5)$$

When formulating the ***gravitation law***, Newton was ***not sure*** that the mass of particles in this law is the same as their ***inertial mass***. To understand this issue, let's start by assuming that the mass parameter in the

[1] Their dimensions are <<< than the distance between them.

gravitational law may be different from $\boldsymbol{m_{in}}$ and note it as $\boldsymbol{m_{gr}}$. Next, let's consider a body in free fall close to earth surface. The earth will exert on it the gravitational force with magnitude $F_{gr} = G\frac{m_{gr}*M_{Earth}}{R^2_{Earth}}$ (6)

Here, we assumed that the body is close to the surface so that its distance form earth center is $\cong R_{Earth}$.

As this is the net force exerted on the body we apply the second law of Newton and get

$$F_{Net} = m_{in}a = F_{gr} = G\frac{m_{gr}*M_{Earth}}{R^2_{Earth}}\text{; So, } m_{in}a = m_{gr} * G\frac{M_{Earth}}{R^2_{Earth}} = m_{gr} * g \text{ because } G\frac{M_{Earth}}{R^2_{Earth}} = g \quad (7)$$

and finally,
$$a = \frac{m_{gr}}{m_{in}} g \quad (8)$$

The measurements show that the acceleration "a" of bodies in free fall close to the earth is equal to that of "g" calculated from expression (7) i.e. g = ***9.8m/s*** if G = $6.67*10^{-11}Nm^2/kg^2$. This means that $\boldsymbol{m_{gr} / m_{in} = 1}$ and $\boldsymbol{m_{in} = m_{gr}}$. So, one can ***confirm that the gravitational mass is the same as the inertial mass***.

- By applying the gravitational law for a mass m=1kg located at a height "*h*" from earth surface, one get

$$\vec{F} = -G\frac{M_{Earth}}{(R_{Earth}+h)^2}\frac{\vec{R}_{Earth}}{R_{Earth}} \equiv \vec{g} \quad (9)$$

So, the *g-vector is equal to the gravitation force exerted by earth on a mass 1kg*. By measuring the force exerted on the mass 1kg in different locations close to earth, one gets a whole *system of g-vectors* (fig.2). ***The totality of these vectors forms the gravitational field of the earth.***

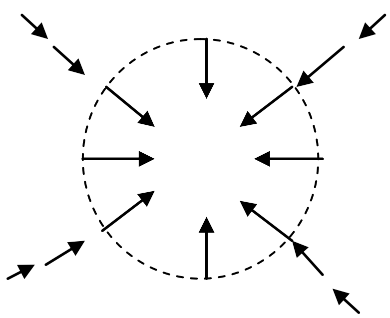

Figure 2 Gravitational field of Earth

Note that the g-vector magnitude decreases with the increase of distance " ***h*** " from the earth but it is always directed versus the center of the earth. The gravitational field of earth has a spherical symmetry. *In fact, it is not exactly spherical, because the model of earth as an uniform density sphere is not accurate*. As the object weight is equal to gravitational force exerted by this field $\vec{F_g} = m\vec{g}$ (10) it comes out that, the weight of the same object is a vector that is different in different points of gravitational field of the earth.

3] KEPLER'S LAWS FOR THE PLANETARY MOTION

- *First law*: *The planets move on elliptic orbits; the sun is located at one of orbits focuses*.

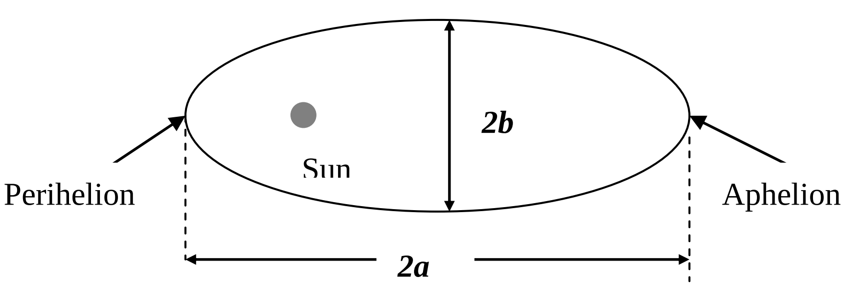

Figure 3 Elliptic orbit of planets

The major and minor axis of the elliptic orbit lengths are labelled respectively as "***2a***" and "***2b***". The edge location of the planet closest to the sun is known as the ***perihelion*** and the edge location at the largest distance to sun is called ***aphelion***.

- *Second law*: *The line sun - planet sweeps out equal areas during equal interval of times*.

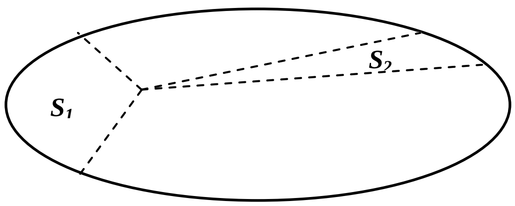

Figure 4 Area "S_1" is equal to area "S_2"

- *Third law*: *The square of the period of planet motion is proportional to the cube of its average distance from the sun*. Calculations show that the **average of distance** sun-planet is equal to half of major axis "***a***". Then, satellite motion (see Lecture_8.5) tells that $\boldsymbol{T^2 = \kappa\, a^3}$ (8)

where $\kappa_{Sun} = \frac{4\pi^2}{GM_{Sun}}$

Note : The three Kepler's laws are valid for elliptical paths of any planet moving around a central body. For example, they apply to the motion of moon around the earth but in this case $\kappa = \kappa_{Earth} = \frac{4\pi^2}{GM_{Earth}}$

THE ENERGY OF A PLANET

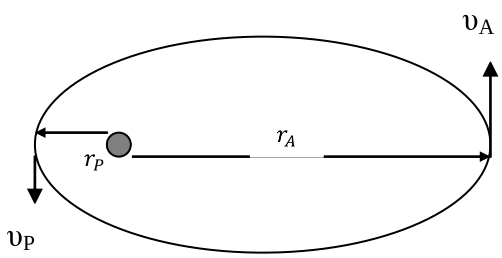

Figure 5

-As the mass of any other planet is much smaller than the mass of sun, one neglects their action on the motion of the studied planet. Then, one considers the *isolated system sun -planet*.
In these circumstances:
a) The net torque of exterior forces is zero and we can apply the ***principle of conservation of angular momentum.***
b) The work done by exterior forces is zero and we can apply the ***principle of energy conservation for the system sun-planet.***

- The principle of angular momentum conservation tells that $\vec{L}_A = \vec{L}_P$ (9) or $\vec{r}_A\, x\, \vec{p}_A = \vec{r}_B\, x\, \vec{p}_P$ (10)
The equality of magnitudes brings to condition $r_A p_A \sin 90^0 = r_P p_P \sin 90^0 \Rightarrow r_A m_{planet} \upsilon_A = r_P m_{planet} \upsilon_P$

So, one get $r_A \upsilon_A = r_B \upsilon_P$ (11)

-The principle of mechanical energy conservation tells that $\boldsymbol{E_A = E_P}$ (12)

As the mechanical energy is E = K + U_G where $K = \frac{m_{pl}}{2} v_{pl}^2$ (13) and $U_G = -G\frac{m_{pl} * M_{Sun}}{r_{pl-Sun}}$ (14)

one get $$\frac{m_{pl}\upsilon_A^2}{2} - G\frac{m_{pl}M_{Sun}}{r_A} = \frac{m_{pl}\upsilon_P^2}{2} - G\frac{m_{pl}M_{Sun}}{r_P} \quad (15)$$

After cancelling m_{pl} ; $\frac{\upsilon_A^2}{2} - G\frac{M_{Sun}}{r_A} = \frac{\upsilon_P^2}{2} - G\frac{M_{Sun}}{r_P}$ and finally $2GM_{Sun}(\frac{1}{r_P} - \frac{1}{r_A}) = \upsilon_P^2 - \upsilon_A^2$ (16)

By using the relation (11) and the fact that $r_A + r_P = 2a$ (17), after some calculations, one find that

$$\upsilon_A^2 = \frac{GM}{a}\frac{r_P}{r_A} \quad (18) \qquad \text{and} \qquad \upsilon_P^2 = \frac{GM}{a}\frac{r_A}{r_P} \quad (19)$$

Then, by substituting one of this expressions at perihelion (or aphelion) i.e. to one side of the expression of total energy (15) one get the ***expression for total mechanical energy of planet*** $E = -G\frac{m_{pl}M_{Sun}}{2a}$ (20)

4] THE BOUND AND UNBOUND TRAJECTORIES

-Let see what happens with an object thrown from the earth vertically up at *initial speed* $\boldsymbol{\upsilon}$. By neglecting the friction, the only force exerted on it is the gravitational force of the earth $F_G = G\frac{m*M_{Earth}}{r^2}$ (21)
As the external net force is zero, the *system earth – object* ***conserves its energy***. For such a system

$\boldsymbol{E = U + K}$ is a constant all time.

Note: *As the reference frame is tied to the earth, generally one talk for mechanical energy of object.*

- By using the work done by the gravitation force (expr.21) one may find[2] that $U(r) = -G\frac{m*M_{Earth}}{r}$ (22)

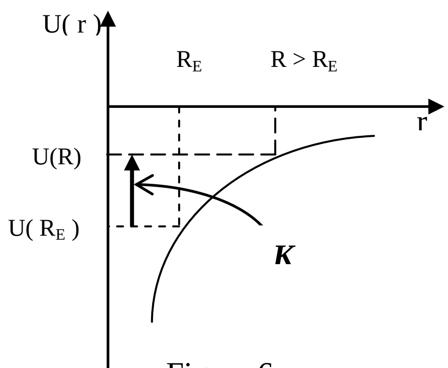

Figure 6

The graph of this function is shown in figure 6. Note that the zero value of potential energy of gravitational interaction is met for big r-values. ***The zero-value of potential energy U means that there is no interaction between the system parts, or more precisely its parts are not bound to each other***. If the object is on the earth surface, the distance from CM of the earth (located at origin of axes in fig.6) is equal to the earth radius R_E and its potential energy is $U(R_E)$. When thrown up vertically at an initial speed $\boldsymbol{\upsilon}$**,** the mechanical energy of the object (or object-earth) at earth surface is

$$\boldsymbol{E(R_E) = U(R_E) + K(\upsilon)} \quad (23)$$

As we know, the ***magnitude of velocity*** will decrease with height and will ***become zero*** at a maximum height $\boldsymbol{h}$, i.e. at a distance $\boldsymbol{R = (R_E + h)}$. When getting at this distance, all the energy is potential energy, So,

$$\boldsymbol{E(R) = U(R)} \quad (24)$$

We say that the initial kinetic energy makes it climb up on a potential well (see fig.6).

Meanwhile, the principle of energy conservation tells that $\boldsymbol{E(R) = E(R_E)}$ (25)

By using this relation we can find the maximum height as follows; by starting from $\boldsymbol{E(R_E+h) = E(R_E)}$

after noting that $\boldsymbol{E(R_E+h) = U(R_E+h)}$ we use relation (23) and get $\boldsymbol{U(R_E+h) = E(R_E) = U(R_E) + K(\upsilon)}$ (26)

So, $-G\frac{mM_{Earth}}{R_E + h} = -G\frac{mM_{Earth}}{R_E} + \frac{m\upsilon^2}{2}$ and $GM_{Earth}(\frac{1}{R_E} - \frac{1}{R_E + h}) = \frac{\upsilon^2}{2}$

Then, $\frac{GM_{Earth}}{R_E}(1 - \frac{1}{1 + h/R_E}) = \frac{\upsilon^2}{2}\_as\_h << R_{E\_} \Rightarrow \frac{GM_{Earth}}{R_E}(1 - 1 + h/R_E) = \frac{\upsilon^2}{2} \Rightarrow \frac{GM_{Earth}}{R_E^2}h = \frac{\upsilon^2}{2}$

As $\frac{GM_{Earth}}{R_e^2} = g$, one may get a result one knows form kinematics of vertical motion $h = \frac{\upsilon^2}{2g}$ (27)

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[2] Through a integral calculus.

- When the **initial speed** υ of the body is such that $U(R_E) + K(\upsilon) < 0$ the object will get to a maximum distance from earth (i.e. a height h_{max}) and will fall back on earth. This means that it will remain ***all time bound*** within the system earth-object;
But, if the initial speed is so big that $U(R_E) + K(\upsilon) \geq 0$ the object will get so far from the earth that the interaction with earth becomes zero and it does not fall back on earth. One says that it becomes an object ***unbounded*** to the gravitational field of the earth. The **minimum initial speed** necessary to unbind from gravitational field of earth is known as ***escape speed*** υ_{esc}. This speed can be found from the condition

$$U(R_E) + K(\upsilon_{esc}) = 0 \Rightarrow -G\frac{mM_E}{R_E} + \frac{m\upsilon_{esc}^2}{2} = 0 \Rightarrow\Rightarrow\Rightarrow\Rightarrow\Rightarrow \upsilon_{esc} = \sqrt{\frac{2GM_E}{R_E}} \quad (28)$$

- Suppose that one must send a rocket out of earth gravitational field; i.e. ***make it unbound*** object to earth. The first requirement is to give to rocket an initial speed $\upsilon \geq \upsilon_{esc}$. A simple calculation based on (28) shows that υ_{esc} **=11190 m/s**. This value of initial speed can be realized only by reactive engines.

What is the shape of the trajectory of an unbound rocket ? The mathematical calculations show that if

a) $\upsilon > \upsilon_{esc}$ the rocket trajectory will ***not be closed*** and it is a ***hyperbola***.
b) $\upsilon = \upsilon_{esc}$ the rocket trajectory will ***not be closed*** and it is a ***parabola***.

If the initial speed of rocket $\upsilon < \upsilon_{esc}$ the object remains a bound object to the earth gravitation field and its ***trajectory*** will be an ***ellipse*** (***closed orbit***). Note that if the initial velocity is "too small" or $\upsilon << \upsilon_{esc}$, this orbit may even cross the earth, i.e. the rocket will strike on earth surface (figure 7).

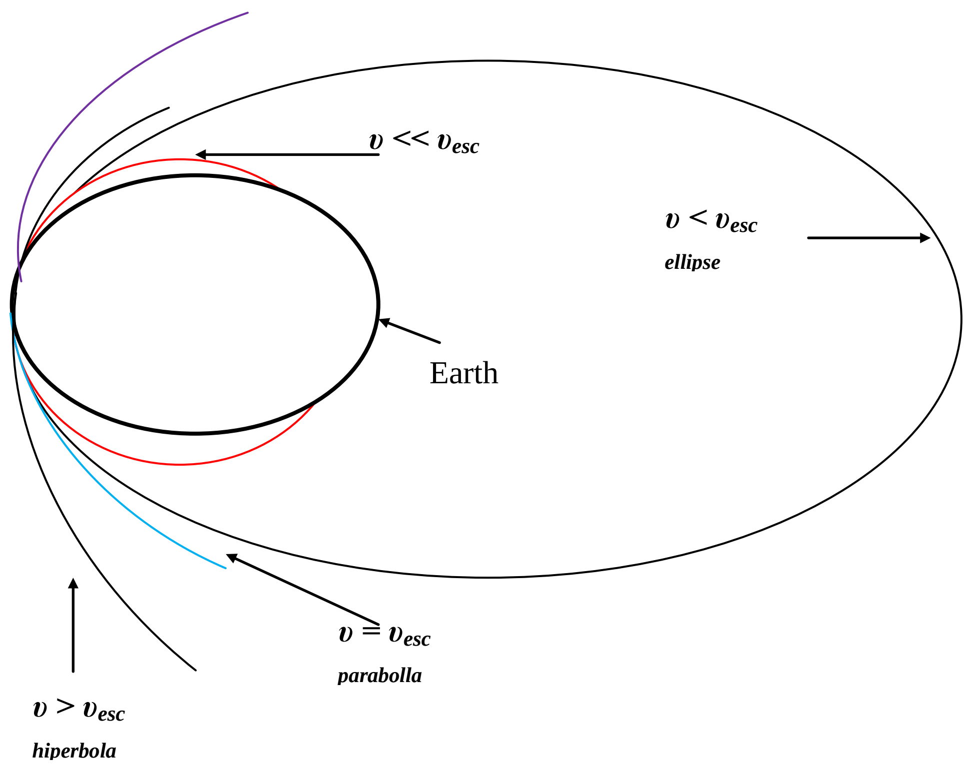

Figure 7 The path shape of a rocket may be hyperbola, parabola or ellipse depending on its initial speed

- For an artificial satellite on a circular orbit at distance R_S from CM of the earth (see fig.8), the acceleration is centripetal $a_c = \frac{\upsilon_s^2}{R_S}$ and the net force exerted on it is equal to the earth gravitation. One can find the magnitude of its constant orbital speed via the following steps:

$$F_{Net} = F_G = G\frac{m_S M_E}{R_S^2} = m_S a = F_C = m_S a_C = m_S \frac{v_S^2}{R_S} \quad \text{i.e.} \quad G\frac{M_E}{R_S^2} = \frac{v_S^2}{R_S} \quad \text{or} \quad v_S^2 = G\frac{M_E}{R_S}$$

So, to rotate at distance R_s from earth center, the satellite must keep an constant speed $v_S = \sqrt{G\frac{M_E}{R_S}}$ (29)

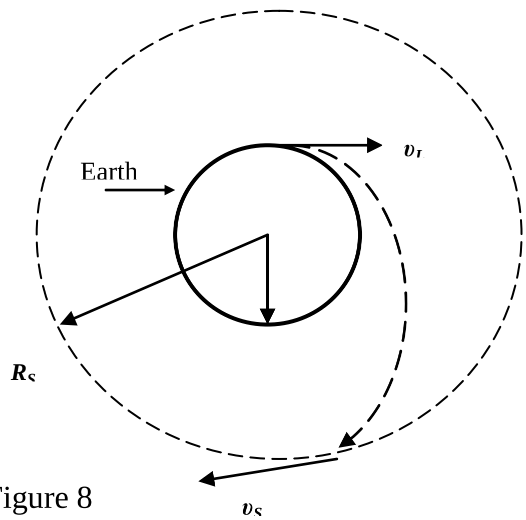

Figure 8

What is the required **launch speed υ_L** for a satellite so that it get to the "distance" R_s from CM of the earth?

One can find the answer of this question by applying the principle of energy conservation and substitute the orbital satellite speed v_S to that given by expression (29).

$$\boldsymbol{E(R_E) = E(R_S)} \quad (30)$$

$$\frac{m_S v_L^2}{2} - G\frac{m_S M_E}{R_E} = \frac{m_S v_S^2}{2} - G\frac{m_S M_E}{R_S} = \frac{m_S}{2} * G\frac{M_E}{R_S} - m_S * G\frac{M_E}{R_S} = -\frac{m_S}{2} * G\frac{M_E}{R_S}$$

By transferring the term ($-G\frac{m_S M_E}{R_E}$) right side and after cancelling " m_S " on both sides, one get

$$\frac{\upsilon_L{}^2}{2} = G\frac{M_E}{R_E} - G\frac{M_E}{2R_S} = G\frac{M_E}{R_S}(\frac{R_S}{R_E} - \frac{1}{2}) \quad (31)$$

and finally

$$\upsilon_L{}^2 = 2G\frac{M_E}{R_S}(\frac{R_S}{R_E} - \frac{1}{2}) = 2G\frac{M_E}{R_S}(\frac{R_E + h}{R_E} - \frac{1}{2}) = \\ = 2G\frac{M_E}{R_S}(1 + \frac{h}{R_E} - \frac{1}{2}) = G\frac{M_E}{R_S}(1 + \frac{2h}{R_E}) \quad (32)$$

For *low orbit* satellites (i.e. heights 100 - 200Km from earth surface), $R_S \approx R_E$(**6378km**), $2h/R_E \approx 0$ and from (32) . we can find that $\upsilon_L^2 \cong G\frac{M_E}{R_S}$.

So, the launch speed for a satellite that rotates in a stationary "low" circular orbit with radius R_S is

$$v_L \cong \sqrt{G\frac{M_E}{R_S}} = \sqrt{G\frac{M_E}{R_E + h}} \quad (33)$$