

LECTURE 2

1. THE SPACE RELATED PROPRIETIES OF PHYSICAL PARAMETERS

Physics uses physical parameters. In this course one will deal only¹ with scalar and vector parameters. Scalar parameters do not depend *on the space direction*. Vector parameters depend on *space directions*.
 Ex: An insect moves on a plan(i.e. a 2D space); its **displacement** is a vector but the **travelled distance** is a scalar.

2. VECTOR NOTATION AND OPERATIONS WITH VECTORS

A vector is shown as a **directed line** and one labels it by a **symbol** (letters) covered by an **arrow line**.

Ex.: \vec{S} or \overrightarrow{AB} for displacement from A to B ... \vec{v} for velocity, ... \vec{F} for force

The *length* of the vector line is proportional to the vector **magnitude**. The magnitude is noted by vector label or its absolute value sign (ex. v or $|\vec{v}|$ for velocity). Note that the *magnitude* itself is a **positive scalar**. The direction of line shows the **direction** of vector *in space* and the **arrow** shows its orientation *sense*.

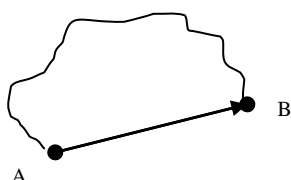


Figure 1

Equal displacement vectors. $\vec{AB} = \vec{CD}$
 Two vectors are equal if they have;
 same **units**, equal **magnitudes** and
 same **orientations** in space.



Figure 2

3. BASIC OPERATIONS WITH VECTORSMultiplying a vector by a scalar(with or without dimensions)

-When **multiplied/divided** by a **positive scalar** without dimensions, **only** the vector's **length** changes.

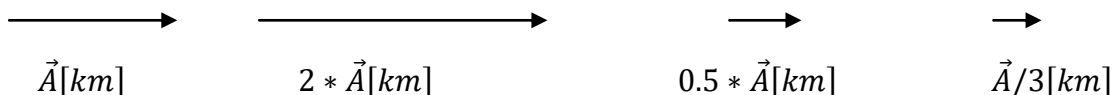


Figure 3 (same units)

-When **multiplied/divided** by a **negative scalar** without dimensions, the vector's **length** changes **and** the orientation is inverted.



Figure 4 (same units)

-If the vector is **multiplied/divided** by a **scalar with dimensions**, the same rules apply for orientation but the new vector has **different units** because it is another **physical quantity** (figure 5).

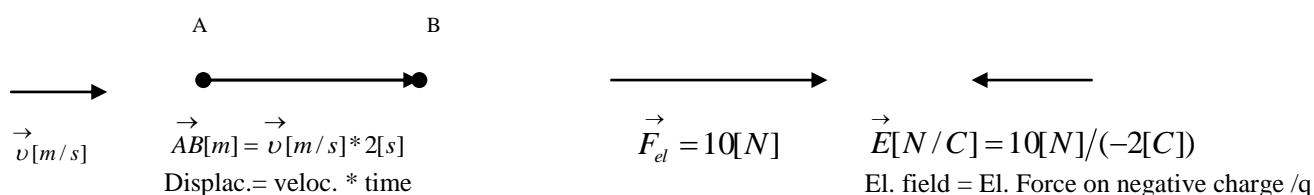


Figure 5 (different units)

¹ Some physical parameters called “tensors” depend in a more complicated way on direction in space.

Addition and subtraction of two vectors

These two operations are **allowed only if** the vectors represent the **same type of physical quantities** i.e. if they have the **same units**. One **cannot add** a **velocity vector** to a **displacement vector**. One applies the **Tip –Tail** method for vector addition; **Shift one of vectors parallel to itself so that its tail fits to the tip of the other vector**. The **vector sum** has the **tail located at the free tail** and the **tip located at the free tip**.

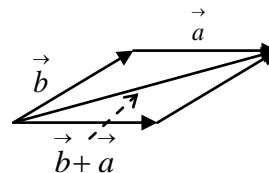
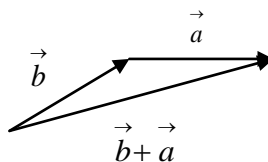
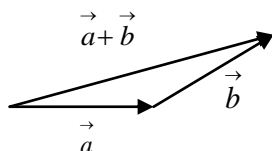
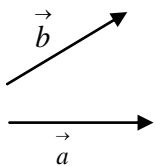


Figure 6.1

6.2

6.3

6.4

By comparing drawings in fig.6.2 and 6.3, 6.4 one can see that the vector $\vec{a} + \vec{b} = \vec{b} + \vec{a}$ (1)

-To subtract \vec{b} from \vec{a} , at first, one multiplies \vec{b} by (-1) and gets the vector $-\vec{b}$ (fig7.2).

Next, one applies the rules for addition of vectors \vec{a} and $-\vec{b}$ (Figure 7.1-4).

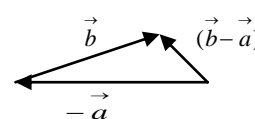
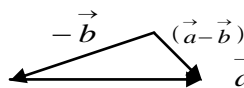
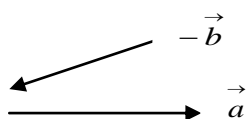
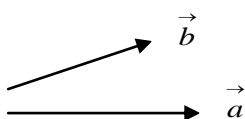


Figure 7.1

(7.2)

(7.3)

(7.4)

By comparing drawings in fig.7.3 and 7.4 one sees that the vector $\vec{a} - \vec{b} = -(\vec{b} - \vec{a})$ (2)

Important note: In general, the magnitude of vector result **is not equal** to the sum or difference of magnitudes of added or subtracted vectors.

$$|\vec{a} + \vec{b}| \neq |\vec{a}| + |\vec{b}| \quad \text{and} \quad |\vec{a} - \vec{b}| \neq |\vec{a}| - |\vec{b}| \quad (3)$$

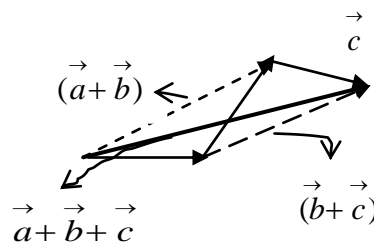
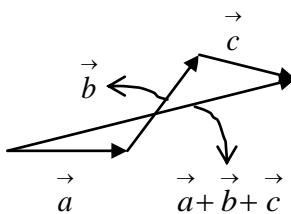
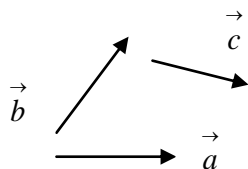


Figure 8.1

8.2

8.3

Note: The **addition of vectors is associative:** $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$ (4)

The figures 8.2 and 8.3 show how to apply the tail-tip rule for the sum of three vectors.

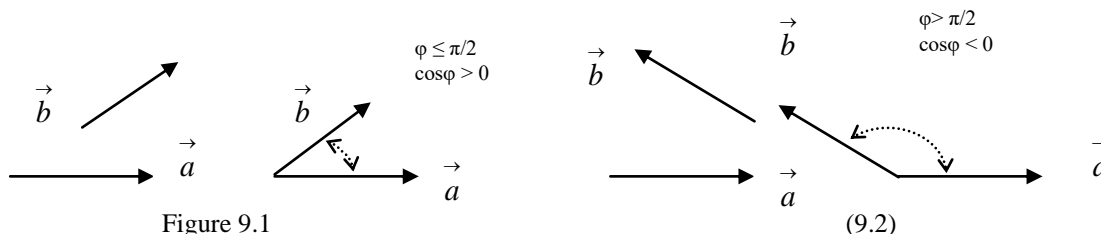
Multiplying two vectors.

-The **scalar (or dot *) product of two vectors** is a **scalar quantity with a new dimension**.

It is defined by the expression

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \phi \quad (5)$$

and can be positive, negative or zero depending on the angle between the vectors (see fig. 9.1-2).



One can show that the **scalar product** is **distributive**:

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} \quad (6)$$

-The **vector(cross) product of two vectors** is a **new vector**, say \vec{C} , noted as $\vec{C} = \vec{A} \times \vec{B}$ (7)

This vector has **magnitude** $|\vec{C}| = |\vec{A}| |\vec{B}| \sin \phi$ (8) ϕ - is the smallest angle from \vec{A} to \vec{B} .

One must refer to the angle $0 \leq \phi \leq \pi$ because the **magnitude of new vector can not be negative**.

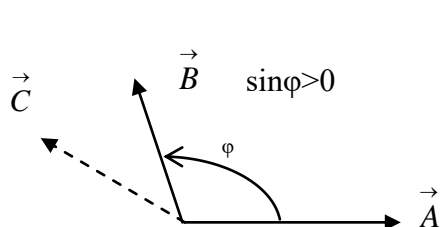


Figure 10

The direction of \vec{C} is perpendicular to the *plane defined by vectors \vec{A} and \vec{B}* ; its **orientation** is defined by convention, using the right-hand rule: *Curl the fingers of the right hand from \vec{A} to \vec{B} over the smallest angle and keep the thumb up. Thumb's direction gives the orientation of \vec{C}* . One may use also the rule of right hand screw;

"The **right-hand screw rule**": If a right-hand screw turns from \vec{A} to \vec{B} along smallest angle between them, the direction of its advancement is that of cross product vector \vec{C} .

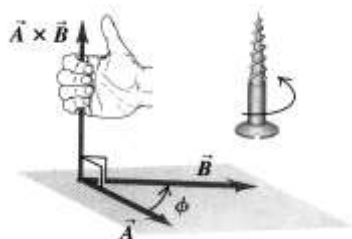
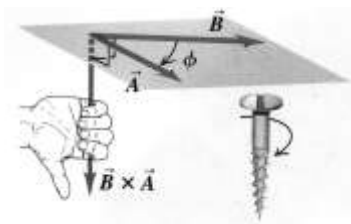


Figure 11.1



(11.2)

As seen from the figure 11.1-2, the **vector product is not commutative**:

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \quad (9)$$

Unit vectors, coordinative systems and vector components.

- **Unit vectors.** A **unit vector** is a **dimensionless** vector with **unit magnitude** that serves only to **define a direction in space**. It is very useful for presentation of different physical vectors. In this course, we will use **right-handed** Cartesian systems (see Fig. 12) on which Ox, Oy, Oz directions are defined by three perpendicular unit vectors $\hat{i}, \hat{j}, \hat{k}$. By using the addition rule of vectors, one can present a vector \vec{A}

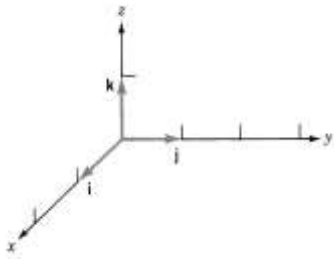


Figure 12

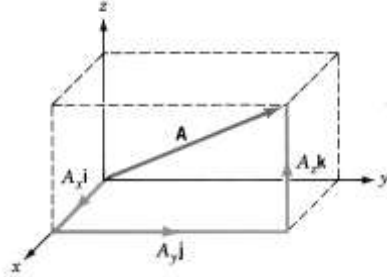


Figure 13

as
$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \quad (10)$$

$A_x \hat{i}$ is a vector along Ox axe with $|A_x|$ magnitude; A_x may be positive or negative,

The unit of the vector \vec{A} goes with its components A_x, A_y, A_z . So, one uses the same set of unit vectors $\hat{i}, \hat{j}, \hat{k}$ to present different vector physical quantities.

Ex: $\vec{v}[\text{m/s}] = 5[\text{m/s}]\hat{i} + 2[\text{m/s}]\hat{j} - 0.4[\text{m/s}]\hat{k}$ and $\vec{F}[\text{N}] = 0.2[\text{N}]\hat{i} + 3.4[\text{N}]\hat{j} + 1.4[\text{N}]\hat{k}$

Note: In general, if one mention the components of vector in a given frame, one means the set of three scalar quantities A_x, A_y, A_z . But, sometimes one refers to the vector components i.e. $\vec{A}_x = A_x \hat{i}$; $\vec{A}_y = A_y \hat{j}$; $\vec{A}_z = A_z \hat{k}$;

- If \vec{A} is a displacement, then $A_x [\text{m}], A_y [\text{m}], A_z [\text{m}]$; if \vec{A} is velocity $A_x [\text{m/s}], A_y [\text{m/s}], A_z [\text{m/s}]$.

The three **components** A_x, A_y, A_z in the reference frame Oxyz define fully the vector \vec{A} . By knowing the three components A_x, A_y, A_z , one may find the vector **magnitude** via Pythagora's theorem as

$$|\vec{A}| \equiv A = \sqrt{A_x^2 + A_y^2 + A_z^2} \quad (11)$$

- One may express the sum or the difference of two vectors through their components as follows:

If $\vec{C} = \vec{A} + \vec{B}$ then $C_x = A_x + B_x$; $C_y = A_y + B_y$; $C_z = A_z + B_z$ (12)

If $\vec{C} = \vec{A} - \vec{B}$ then $C_x = A_x - B_x$; $C_y = A_y - B_y$; $C_z = A_z - B_z$ (13)

If $\vec{C} = \vec{A}$ then $C_x = A_x$; $C_y = A_y$; $C_z = A_z$ (14)

The **unit vector** \hat{A} along the direction of \vec{A} has the **same** space **direction** as vector \vec{A} , the **magnitude "1"** but it has **no dimension**. So, by using (10) and (11) one gets

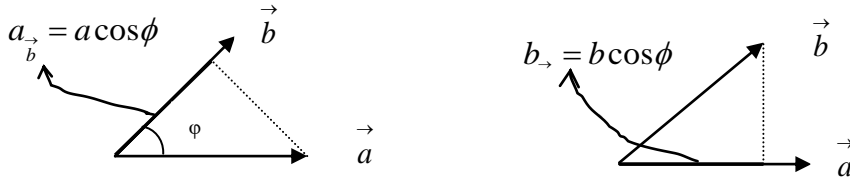
$$\hat{A} = \frac{\vec{A}}{|\vec{A}|} = \frac{\vec{A}}{\sqrt{A_x^2 + A_y^2 + A_z^2}} = \frac{A_x \hat{i} + A_y \hat{j} + A_z \hat{k}}{\sqrt{A_x^2 + A_y^2 + A_z^2}} \quad (15)$$

Ex: If $\vec{a} = 1\text{m}\hat{i} + 2\text{m}\hat{j} + 3\text{m}\hat{k}$ then $\hat{a} = \frac{1\text{m}\hat{i} + 2\text{m}\hat{j} + 3\text{m}\hat{k}}{\sqrt{1^2 + 2^2 + 3^2}\text{m}} = \frac{1\hat{i} + 2\hat{j} + 3\hat{k}}{\sqrt{14}} = 0.267\hat{i} + 0.535\hat{j} + 0.802\hat{k}$

Expressing the dot and cross product by use of vector components in a Cartesian system Oxyz

-Dot product (*) $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \phi = ab \cos \phi = (a_{\vec{b}})b = a(b_{\vec{a}})$ (16)

Figure 14



One may figure out that if α, β, γ are the angles of a vector \vec{A} to axes Ox, Oy, Oz, then one may calculate the scalar **components of vector** \vec{A} along these axes as:

$$A_x = A \cos \alpha; \quad A_y = A \cos \beta; \quad A_z = A \cos \gamma \quad (17)$$

Note that the dot product between each two unit vectors of a Cartesian frame Oxyz, gives or 1 or 0.

$$\begin{aligned} \hat{i} \cdot \hat{i} &= 1 \cdot 1 \cdot \cos 0 = 1; & \hat{i} \cdot \hat{j} &= 1 \cdot 1 \cdot \cos \pi/2 = 0; & \hat{i} \cdot \hat{k} &= 1 \cdot 1 \cdot \cos \pi/2 = 0 \\ \hat{j} \cdot \hat{j} &= 1 \cdot 1 \cdot \cos 0 = 1; & \hat{j} \cdot \hat{i} &= 1 \cdot 1 \cdot \cos \pi/2 = 0; & \hat{j} \cdot \hat{k} &= 1 \cdot 1 \cdot \cos \pi/2 = 0 \\ \hat{k} \cdot \hat{k} &= 1 \cdot 1 \cdot \cos 0 = 1; & \hat{k} \cdot \hat{i} &= 1 \cdot 1 \cdot \cos \pi/2 = 0; & \hat{k} \cdot \hat{j} &= 1 \cdot 1 \cdot \cos \pi/2 = 0 \end{aligned} \quad (18)$$

Consider two vectors $\vec{A}(A_x, A_y, A_z)$ and $\vec{B}(B_x, B_y, B_z)$ and their components in the same Oxyz frame.

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \quad \text{and} \quad \vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k} \quad (19)$$

One may calculate the dot product $\vec{A} \cdot \vec{B}$ by using the components (19) in the system of axes Oxyz. By applying the results of dot product between unit vectors (18), one gets

$$\vec{A} \cdot \vec{B} = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \cdot (B_x \hat{i} + B_y \hat{j} + B_z \hat{k}) = A_x B_x + A_y B_y + A_z B_z \quad (20)$$

Important example: Calculate the **magnitude** of \vec{A} by its components (A_x, A_y, A_z) in a frame Oxyz.

$$\text{As } \vec{A} \cdot \vec{A} = A \cdot A \cos 0 = A^2 \quad (*) \quad \text{and} \quad \vec{A} \cdot \vec{A} = A_x A_x + A_y A_y + A_z A_z = A_x^2 + A_y^2 + A_z^2 \quad (**)$$

$$\text{by comparing } (*) \text{ and } (**) \text{ one finds out that } A^2 = A_x^2 + A_y^2 + A_z^2 \text{ and } A = \sqrt{A^2} = \sqrt{A_x^2 + A_y^2 + A_z^2} \quad (21)$$

-Cross product "x" After introducing the **concept of unit vector** one can define the cross product by expression

$$\vec{A} \times \vec{B} = (AB \sin \phi) \hat{n} \quad (22)$$

where \hat{n} is a **unit vector perpendicular to the plane** defined by vectors \vec{A}, \vec{B} . Also, one may find the **components of the vector** $(\vec{A} \times \vec{B})$ by their components; i.e. if $\vec{A}(A_x, A_y, A_z)$ and $\vec{B}(B_x, B_y, B_z)$

$$\begin{aligned} \vec{A} \times \vec{B} &= (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \times (B_x \hat{i} + B_y \hat{j} + B_z \hat{k}) = \left\{ A_x B_x (\hat{i} \times \hat{i}) + A_x B_y (\hat{i} \times \hat{j}) + A_x B_z (\hat{i} \times \hat{k}) \right\} + \\ &\left\{ A_y B_x (\hat{j} \times \hat{i}) + A_y B_y (\hat{j} \times \hat{j}) + A_y B_z (\hat{j} \times \hat{k}) \right\} + \left\{ A_z B_x (\hat{k} \times \hat{i}) + A_z B_y (\hat{k} \times \hat{j}) + A_z B_z (\hat{k} \times \hat{k}) \right\} \end{aligned} \quad (23)$$

As $\sin 0^\circ = 0$ and $\sin \pi/2 = 1$, when applied between the unit vectors, the expressions (22) gives:

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 1 \sin 0 = 0 \quad \text{and} \quad \hat{i} \times \hat{j} = (1 \sin \pi/2) \hat{k} = \hat{k}; \quad \hat{i} \times \hat{k} = -\hat{j}; \quad \hat{j} \times \hat{k} = \hat{i} \quad \dots \quad (24)$$

Taking into account the relations (24) one finds out that the expression (23) transforms to

$$\begin{aligned}\vec{A} \times \vec{B} &= \left\{ A_x B_y (\hat{i} \times \hat{j}) + A_x B_z (\hat{i} \times \hat{k}) \right\} + \left\{ A_y B_x (\hat{j} \times \hat{i}) + A_y B_z (\hat{j} \times \hat{k}) \right\} + \left\{ A_z B_x (\hat{k} \times \hat{i}) + A_z B_y (\hat{k} \times \hat{j}) \right\} \\ &= \left\{ A_x B_y \hat{k} - A_x B_z \hat{j} \right\} + \left\{ -A_y B_x \hat{k} + A_y B_z \hat{i} \right\} + \left\{ A_z B_x \hat{j} - A_z B_y \hat{i} \right\} = \\ &= (A_y B_z - A_z B_y) \hat{i} + (A_z B_x - A_x B_z) \hat{j} + (A_x B_y - A_y B_x) \hat{k}\end{aligned}\quad (25)$$

So, if $\vec{A} \times \vec{B} \equiv \vec{C}(C_x, C_y, C_z)$ then

$$C_x = A_y B_z - A_z B_y; \quad C_y = A_z B_x - A_x B_z; \quad C_z = A_x B_y - A_y B_x \quad (26)$$

Note: One may get quickly the components of cross product through the following determinant

$$\vec{A} \times \vec{B} \equiv \vec{C} = C_x \hat{i} + C_y \hat{j} + C_z \hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad (27)$$

The application of vector algebra in translational & rotational equilibrium

- One says that a **force "F" is applied on an object** " if **this object undergoes an action** that **tends**:

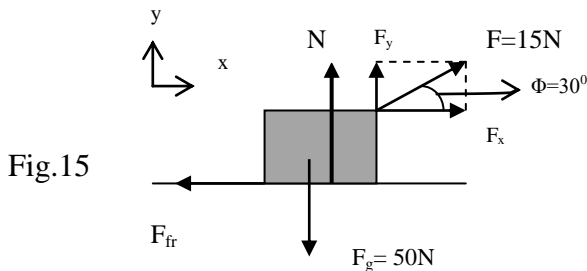
- 1] **to move it** along a given space direction, if the object is *at rest*; or
- 2] **to change its velocity vector** (magnitude, direction or both), if the object is in movement.

If the object **velocity does not change** even though several forces are acting on it, one says that it is in **translational equilibrium**. In this case, the sum of all forces acting on the object is zero.

$$\sum_i \vec{F}_i = 0 \quad (28)$$

Example#1: A 5kg brick remains at rest ($v=0$) on a table even though one applies a 15N force directed at 30° above the horizontal (fig.15). Find the *magnitudes* of *friction* and *normal* forces on the brick.

Note: The **friction** force is exerted by the table on the brick along its surface of contact. It has **opposite direction** to the direction of possible block shift. Fig. 15 shows the set of all forces acting on the brick.



There are four forces **exerted on the brick**: $\vec{F}_g; \vec{N}; \vec{F}; \vec{F}_{fr}$.

As the **brick is at rest** one writes the relation (28) as

$$\sum_i \vec{F}_i = \vec{F}_g + \vec{N} + \vec{F} + \vec{F}_{fr} = 0 \quad (29)$$

Next, one selects axes **Ox, Oy** as shown and project the vector eq.(29) on each of them. ($F_x = 15N \cos 30^\circ$ and $F_y = 15N \sin 30^\circ$)

$$(\vec{F}_g = F_g * (-\hat{i}); \quad \vec{N} = N * (\hat{j}); \quad \vec{F} = 15 \cos 30^\circ (\hat{i}) + 15 \sin 30^\circ (\hat{j}); \quad \vec{F}_{fr} = F_{fr} * (-\hat{i})$$

$$\text{Ox: } F_x - F_{fr} = 15 \cos 30^\circ - F_{fr} = 0 \rightarrow F_{fr} = 15 \cos 30^\circ = 13.0N$$

$$\text{Oy: } N - F_g + F_y = N - 5 * 10 + 15 \sin 30^\circ = 0 \rightarrow N = 50 - 15 \sin 30^\circ = 42.5N \quad (\text{taking } g = 10m/s^2)$$

Example#2: Let's consider a bolt that may rotate around a **fixed axis** passing through its center (axe Oz in fig.16). Let's assume that the exerted action is not enough to start the rotation. So, it remains at rest. One says that this is an object in the state of **rotational equilibrium**.

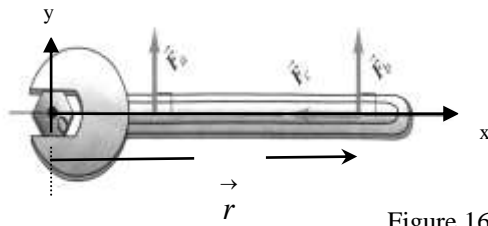


Figure 16

From our experience, we know that the **rotational action** (we call it **torque** action) on the bolt is larger if :

- the **magnitude** of exerted force F is larger.
- the force is applied at larger **distance from rotation axis**.
- the force is applied at 90° **angle** to wrench axis.

So, the **rotation action** is $\sim \left| \vec{F} \right|, \sim \left| \vec{r} \right|, \sim \sin(\vec{r}, \vec{F})$. One defines the **torque vector** as $\vec{\tau} = \vec{r} \times \vec{F}$ (30)

- The **torque** definition by the **cross product** does make sense: its magnitude is $\left| \vec{\tau} \right| = \left| \vec{r} \right| \left| \vec{F} \right| \sin(\vec{r}, \vec{F})$

and it gives the orientation of rotational action, too. In bolt example (fig.16), the torque is directed along Oz ($\vec{r}_{\text{along } \hat{i}}; \vec{F}_{\text{along } \hat{j}} \text{ and } \vec{\tau}_{\text{along } \hat{k}} = \hat{i} \times \hat{j}$) and rotational action is directed CCW.

Important note: The torque is defined always with respect to a specific point "O": an axe of possible rotation passes by this point. When talking about the torque, one must precise the torque with respect to "O-point". In rotation problems, one places the origin of coordinate system at O point.

-As we will see, in **rotational equilibrium**, the sum of all torques acting on the object is equal to zero.

$$\sum_i \vec{\tau}_i = 0 \quad (31)$$

By applying the condition (31) in the case of **bolt** at rotational equilibrium, we find that the action of the **torque** applied by **wrench** is cancelled by that of an "internal" **torque** due to friction of wall on the bolt.

These two torques have equal magnitude but opposite direction $\vec{\tau}_{\text{ext}} + \vec{\tau}_{\text{int}} = 0 \Rightarrow \vec{\tau}_{\text{int}} = -\vec{\tau}_{\text{ext}}$ (32)

Exemple: Find the **components of applied torque on wrench** if $\left| \vec{r} \right| = 30\text{cm} = 0.3\text{m}$ and the applied force

15N is directed along Oy (or \hat{j}). So, in Oxy system (see fig.16) the applied force has the components (0, 15N, 0) and the position r -vector has components (0.3m, 0m, 0m). By using the cross product general formula one gets

$$\begin{aligned} \vec{\tau} &= \tau_x \hat{i} + \tau_y \hat{j} + \tau_z \hat{k} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0.3\text{m} & 0 & 0 \\ 0 & 15\text{N} & 0 \end{vmatrix} \\ &= \hat{i}(0 * 0 - 0 * 15\text{N}) - \hat{j}(0.3\text{m} * 0 - 0 * 0) + \hat{k}(0.3\text{m} * 15\text{N} - 0 * 0) \end{aligned}$$

So, $\tau_x = 0; \tau_y = 0; \tau_z = 4.5\text{Nm}$. The same result comes out by using the definition of magnitude by using

the basic relation $\left| \vec{\tau} \right| = \left| \vec{r} \right| \left| \vec{F} \right| \sin(\vec{r}, \vec{F}) = 0.3\text{m} \cdot 15\text{N} \cdot \sin(90^\circ) = 4.5\text{Nm}$.

One may get the same magnitude of torque even by applying a **force** with **smaller** magnitude but bigger $\left| \vec{r} \right|$ (**bigger lever arm**). (Ex: Find F magnitude that applies the same torque if $\left| \vec{r} \right| = 0.4\text{m}$).