K.Angoni, Physics Department, Vanier College

LECTURE 4

1] AREA UNDER THE KINEMATIC'S GRAPHS

- We saw how to find ***instantaneous velocity*** from the graph ***x = x(t)*** and ***instantaneous acceleration*** from the graph $\upsilon = \upsilon(t)$ by using the graph slope *(or derivatives* **dx/dt; dυ/dt**) at the point of interest.
The following " inverse " situations are met in practice, too:
a) Given the graph $\upsilon = \upsilon(t)$, find the location of particle "***x at a given moment t"***.
b) Given the graph $a = a(t)$, find the velocity of particle "***υ at a given moment t"***.

A] Finding the location of particle from the velocity graph υ = υ(t)

- The expression $\upsilon = dx/dt$ ***(1)*** *is a pure mathematical definition.* In physics, when referring to s***mall*** *but* ***finite*** *and* **measurable** intervals (Δx or Δt), one writes expression (1) as $\upsilon = \Delta x/\Delta t$ (2)
So, for small intervals (Δx or Δt) in real measurements, one gets $\Delta x = \upsilon * \Delta t$ (3)
Then, for a motion at constant velocity (graph in fig. 1) *one may calculate the* ***displacement*** Δx *from an initial location* X_I *(at time* t_I*) as* $\Delta x = X_F - X_I = \upsilon * \Delta t = \upsilon * (t_F - t_I)$ (4)
As $\upsilon * (t_F - t_I)$ ***is the area of rectangle (fig.1), the displacement is equal to the area under the velocity graph for the considered time interval.*** Finally, one gets $X_F = X_I + \Delta x = X_I + \upsilon * (t_F - t_I)$ (5)
Note that one must know X_I so that one can find X_F by expression (5).

Figure 1

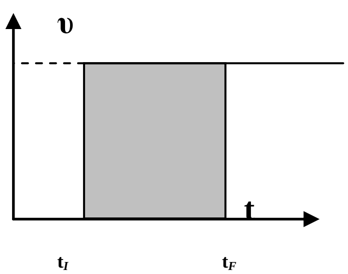
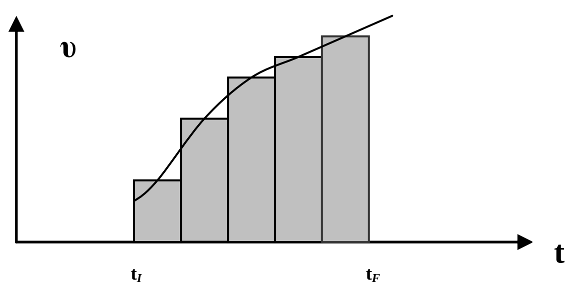


Figure 2



- The result "***displacement*** *(*$\Delta x = X_F - X_I$*)* ***equal to area under velocity graph***" *is valid in all cases* but the way ones calculates the area must be modified if the velocity is not constant (see fig.2). One has to calculate the area by using a set of narrow rectangles covering the better way possible the area under the graph. One may figure out that this method is more accurate[1] when using smaller time intervals Δt.

- The figures 3,4 show two possible graphs of velocity. As the area under graph in fig.3 is a positive value, it comes out that $\Delta x = X_F - X_I > 0$ and $X_F > X_I$. This means that the particle is moving along ***positive direction*** of ***Ox***. In figure 4, the area under graph is negative, so one has $\Delta x = X_F - X_I < 0$ and $X_F < X_I$. This means that the particle is moving along ***negative direction*** of ***Ox***.

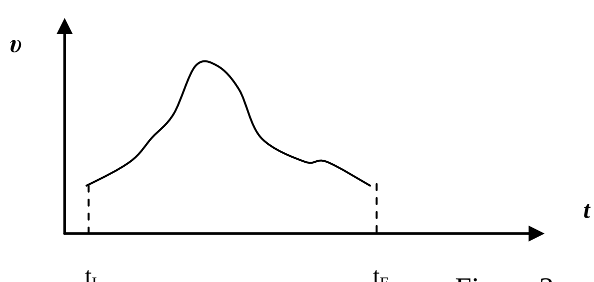


Figure 3

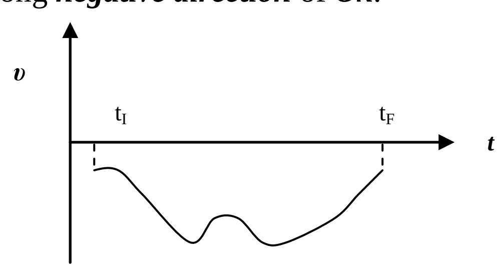


Figure 4

[1] The most accurate calculation is done by using the integral calculation.

B] Finding the velocity of particle from the acceleration graph a=a(t)

- Similarly, if one has the graph $a = a(t)$, one may calculate the ***change of*** particle ***velocity*** by the area under the graph. In the case of motion at ***constant acceleration*** (figure 5.a,b), one may easily calculate the velocity at any moment " υ_F "(*F*-final) if one knows it at an initial " υ_I " moment of time.

The physic's definition for acceleration is $a = \Delta\upsilon / \Delta t$ (6)

For constant acceleration one may write $a = (\upsilon_F - \upsilon_I) / (t_F - t_I)$ (7)

So, $\upsilon_F - \upsilon_I = a*(t_F - t_I)$ *and* $\upsilon_F = \upsilon_I + a*(t_F - t_I)$ (8)

As time change is positive, ***a-sign*** decides if the velocity increases (5.a) or decreases (5.b) with time.

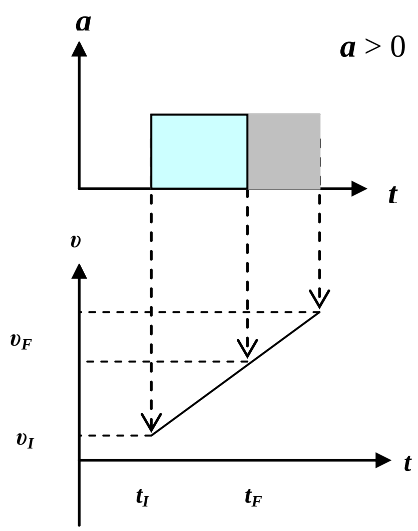
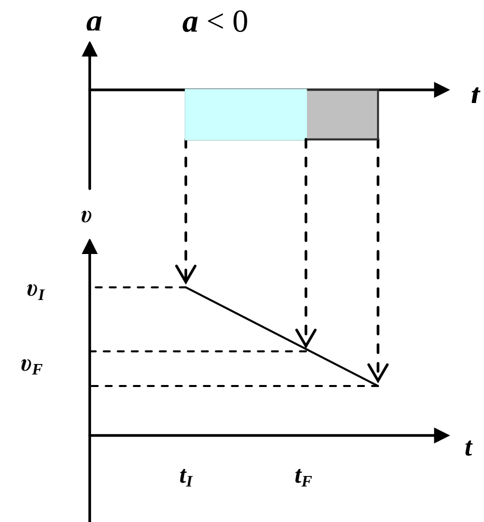


Figure 5.a (a-c^te, positive, υ-increases linearly)



5.b (a-c^te, negative, υ-decreases linearly)

Even when "***a***" is not constant, the area under the a(t) graph gives the *change of velocity* but in these cases the velocity does not change linearly with time (Ex. the "harmonic oscillations, see NYC course).

2] BASIC KINEMATIC'S EQUATIONS OF 1D MOTION AT ***CONSTANT ACCELERATION***

- Finding the velocity by the slope of graph $x = x(t)$ or the displacement by the area under graph $\upsilon = \upsilon(t)$ is useful, but when the graphs are not straight lines and one looks for instantaneous velocity at several time moments, the procedure becomes cumbersome and complicated. In those situations, the analytical methods provide the result in the form of equations and their solutions as functions of time. By using a function one can find quickly the numerical value of kinematic parameters at any moment of time.

- Let's consider a 1D motion at ***constant acceleration*** "***a***". We know that in this case the velocity is given by expression (8). To simplify expressions, one takes time $t_I \equiv 0$ at the starting moment and use the notations:

$x_I \equiv x_o;\ t_F = t;\ x_F \equiv x;\ \upsilon_I \equiv \upsilon_o$ and $\upsilon_F \equiv \upsilon$ (9)

With these notations the expression (8) can be written as

$$\upsilon = \upsilon_0 + a*t \qquad (10)$$

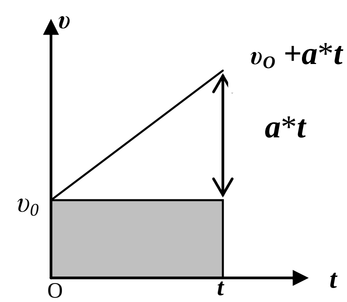


Figure 6

The inclined line in fig.6 is described by expression (10).
Its slope is equal to "***a***".

- Then, one can find the displacement as the area under the velocity graph in figure 6.

This area is constituted by two parts: the rectangular area $\boldsymbol{A_1 = \upsilon_{O} * t}$ (11)

and the triangle with area $\boldsymbol{A_2 = t*at/2 = a*t^2/2}$ (12)

So, the total area under the velocity graph is $\boldsymbol{A_{tot} = A_1 + A_2 = \upsilon_{O} * t + a*t^2/2}$ (13)

This way, one finds out that $\Delta x \equiv A_{tot} = x - x_0 = \upsilon_0 * t + a\frac{t^2}{2}$ and $x = x_0 + \upsilon_0 * t + a\frac{t^2}{2}$ (14)

Note: The graph x = x(t) of expression (14) is a parabola which, at t = 0s, crosses *x-axe* at $\boldsymbol{x_0}$ and has a slope (at t=0 sec) equal to $\boldsymbol{\upsilon_0}$. For different values of ***x_0 and υ_0*** *this parabola has different orientations in **x-t** plane. A motion at constant accelerations is presented always by a parabola in **x-t** plane.*

- The expression (14) can be modified so that it relates the velocity and position without the variable "time". One may start by isolating "**t**" at expression (10) $t = \frac{\upsilon - \upsilon_0}{a}$ (15)

Next, by substituting it at expression (14) one gets:

$$x = x_0 + \upsilon_0 * \left(\frac{\upsilon - \upsilon_0}{a}\right) + \frac{a}{2}\left(\frac{\upsilon - \upsilon_0}{a}\right)^2 \Rightarrow 2a(x - x_0) = 2\upsilon_0 * (\upsilon - \upsilon_0) + (\upsilon - \upsilon_0)^2 = 2\upsilon_0\upsilon - 2\upsilon_0^2 + \upsilon^2 - 2\upsilon\upsilon_0 + \upsilon_0^2$$

$2a(x - x_0) = \upsilon^2 - \upsilon_0^2$ and finally $\upsilon^2 = \upsilon_0^2 + 2a(x - x_0)$ (16)

-The three equations

$$x = x_0 + \upsilon_0 * t + a\frac{t^2}{2}$$
$$\upsilon = \upsilon_0 + a * t \qquad (17)$$
$$\upsilon^2 = \upsilon_0^2 + 2a(x - x_0)$$

are the ***three basic kinematics equations*** for the 1D motion at ***<u>constant acceleration</u>***.

Note: *Some textbooks write the first expression as* $\Delta x = x - x_0 = v_0 * t + a\frac{t^2}{2}$ *and relate the calculations to the displacement "* $\boldsymbol{\Delta x}$*". As this requests strong attention to the sign of Δx, it is better to use these expressions in the form given at (17) and derive the displacement **Δx** if the problem asks for it.*

-By transforming eq. (16) as follows $\upsilon^2 - \upsilon_0^2 = 2 * \frac{\Delta \upsilon}{\Delta t}(x - x_0) \rightarrow (\upsilon - \upsilon_0)(\upsilon + \upsilon_0) = 2 * \frac{\upsilon - \upsilon_0}{t - 0}(x - x_0)$

one gets $x - x_0 = \frac{\upsilon + \upsilon_0}{2} t$ (18)

This expression is useful if one knows that the ***<u>acceleration is constant</u>***, but does not know " ***a-value*** ".

3] FREE FALL

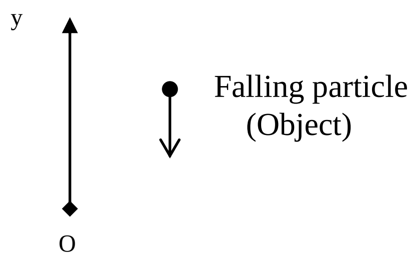
- One say that an object is in a ***free-fall*** *motion if the* ***gravity is the only force*** *acting* on it**.**
A common[2] situation of free fall motion is that of an object falling vertically from a certain height.
The acceleration of ***free-fall motion*** near to earth surface ***is due to earth gravity*** and $a_g \equiv g \approx 9.8 m/s^2$.

- In the case of free-fall "*earth objects*" one has to deal with ***motion along a fixed direction*** (translation) in space and one ***models the object motion*** as the ***motion*** of a ***point*** in 1D Kinematics.
Furthermore, as the acceleration is constant ($9.8 m/s^2 \equiv g$), one may apply the kinematics equations for the motion at constant acceleration.

- When studying the two dimensional kinematics, in general, one uses the ***Ox*** axe for horizontal motion and ***Oy*** axe is directed ***vertically up***. Based on this choice of Oy axe, the *free fall acceleration* will be

$$\boldsymbol{a_y = -g} \qquad (19)$$

- So, for a free-fall motion of objects near earth, the kinematics equations take the form



$$\upsilon = \upsilon_0 - g * t \qquad (20)$$

$$y = y_0 + \upsilon_0 * t - g\frac{t^2}{2} \qquad (21)$$

$$\upsilon^2 = \upsilon_0^2 - 2g(y - y_0) \qquad (22)$$

- The value $g = 9.8 m/s^2$ is *a good approximation* for the magnitude of acceleration of objects in free fall (no other forces but the earth gravitation) ***near to earth surface***. In fact, the magnitude of acceleration due to earth gravity does depend on the altitude and location of object. Note that the free-fall motion is ***not a good model*** for any dropped objects. For example, in the case of skydivers, the action of air friction force, which is directed against the direction of motion, does not allow the gravitation to increase the speed at constant acceleration. One say that the presence of ***air friction force*** destroys the condition of a free-fall motion.

Use the following link to generate 1D kinematics graphs:
https://phet.colorado.edu/sims/cheerpj/moving-man/latest/moving-man.html?simulation=moving-man

[2] Note that one neglects the air friction force when applying the *free fall model* for terrestrial objects.
The motion of a satellite around the earth is an important case of free fall motion.