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LECTURE 5

1] DESCRIPTION OF PARTICLE MOTION IN 2D&3D SPACE

-The position, the displacement, the velocity and the acceleration reveal their vector nature in 1-D motion through their sign but their full vector meaning shows up when the particle is moving in 2D or 3D space. Here one has to deal with their components along the selected axes. Let's consider the motion of a particle from point P_1 to P_2 in 3D space and assume that one has selected the frame ***Oxyz*** with unit vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ shown in fig.1. The location of points P_1, P_2 in this frame is defined by the ***position vectors*** $\vec{r}_1$, $\vec{r}_2$. The *coordinates* of P_1 (***x_1, y_1, z_1***) and P_2 (***x_2, y_2, z_2***) are the *scalar components of* $\vec{r}_1$ and $\vec{r}_2$ in frame Oxyz:

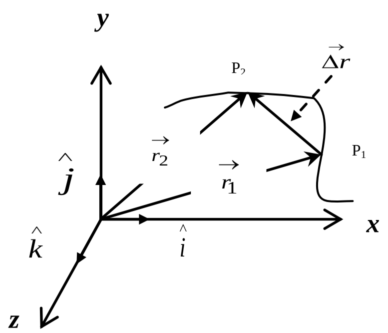

Figure 1

$$\vec{r}_1 = x_1\,\hat{i} + y_1\,\hat{j} + z_1\,\hat{k} \quad \text{and} \quad \vec{r}_2 = x_2\,\hat{i} + y_2\,\hat{j} + z_2\,\hat{k} \qquad (1)$$

One defines the ***displacement vector*** (*from P_1 to P_2*) as

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1 \qquad (2)$$

The *components* of *displacemnent vector* are (from 1 and 2)

$$\Delta\vec{r} = \Delta x\,\hat{i} + \Delta y\,\hat{j} + \Delta z\,\hat{k} \qquad (3)$$

$$\Delta x = x_2 - x_1;\ \Delta y = y_2 - y_1;\ \Delta z = z_2 - z_1 \qquad (4)$$

Note: Often, one uses the symbol $\vec{S}$ for displacement (*instead of $\Delta\vec{r}$*). ***In this case "S" is the magnitude of displacement*** (*not the length of travelled distance)*.

- One defines the *average velocity vector* through the *displacement vector*

$$\vec{\upsilon}_{av} = \frac{\Delta\vec{r}}{\Delta t} \qquad (5)$$

and the corresponding *interval of time Δt*. If the *time interval* becomes small, $\Delta\vec{r}$ direction tents to the tangent of path at starting point (P_1). So, one gets the ***instantaneous velocity at initial point*** defined as

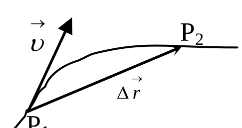

Figure 2

$$\vec{\upsilon}\big|_{P1} = \lim_{\Delta t\to 0}\frac{\Delta\vec{r}}{\Delta t} \equiv \frac{d\vec{r}}{dt}\Big|_{P1} \qquad (6)$$

This vector of velocity is tangent to the path shape at point P_1, but note that its *magnitude* ***is not equal to the slope of the path at that point*** *(because* <u>this is not</u> *a graph*[1] *of position as function of time).*

Being a vector, the *instantaneous velocity* can be expressed by its three components (**υ_x, υ_y, υ_z**) in *Oxyz;*

$$\vec{\upsilon} = \upsilon_x\,\hat{i} + \upsilon_y\,\hat{j} + \upsilon_z\,\hat{k} \qquad (7)$$

As $$\vec{\upsilon} = \frac{d\vec{r}}{dt} = \frac{d(x\,\hat{i} + y\,\hat{j} + z\,\hat{k})}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} \qquad \upsilon_x = \frac{dx}{dt};\ \upsilon_y = \frac{dy}{dt};\ \upsilon_z = \frac{dz}{dt}; \qquad (8)$$

[1] It is a graph that relates different coordinates (x,y,z).

- In a similar way, one would define the *average acceleration* as $\vec{a}_{av} = \frac{\Delta\vec{v}}{\Delta t}$ through the *difference of velocity vectors* at points P_1 and P_2 (see figure 3). Next, by decreasing the *interval of time* Δt one would get to the ***instantaneous acceleration vector*** at point P_1:

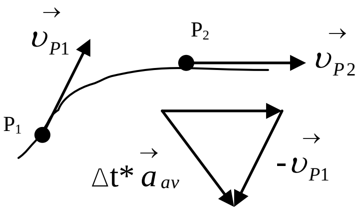

Figure 3

$$\vec{a}\Big|_{P1} = \lim_{\Delta t \to 0} \frac{\Delta \vec{\upsilon}}{\Delta t} = \frac{d\vec{\upsilon}}{dt}\Big|_{P1} \qquad (9)$$

The acceleration vector has three components ($\boldsymbol{a_x}$, $\boldsymbol{a_y}$, $\boldsymbol{a_z}$) in Oxyz

As $$\vec{a} = \frac{d\vec{\upsilon}}{dt} = \frac{d\upsilon_x}{dt}\hat{i} + \frac{d\upsilon_y}{dt}\hat{j} + \frac{d\upsilon_z}{dt}\hat{k} \qquad (10)$$

one get $$a_x = \frac{d\upsilon_x}{dt};\_a_y = \frac{d\upsilon_y}{dt};\_a_z = \frac{d\upsilon_z}{dt}; \qquad (11)$$

and by using (8) $$a_x = \frac{d^2x}{dt^2};\_a_y = \frac{d^2y}{dt^2};\_a_z = \frac{d^2z}{dt^2}; \qquad (12)$$

- If a particle is moving on a plane (*2D space*) ***at constant acceleration,*** $\boldsymbol{\vec{a}_{P1} = \vec{a}_{P2} = \vec{a}}$. Then, from (9) one get $d\vec{\upsilon} = \vec{a}\,dt$ which, for a small but measurable (*physical*) change, is written as $\Delta\vec{\upsilon} = \vec{a}\,\Delta t$.
By taking $t = 0$ when the particle is at P_1, when it gets at P_2 $\Delta t \equiv t$ and $\vec{v}_{P\_1} \equiv \vec{v}_0$, $\vec{v}_{P\_2} \equiv \vec{v}$.

With these new notations one gets the expression $\Delta\vec{v} = \vec{v} - \vec{v}_0 = \vec{a}\cdot t$ or $\vec{\upsilon} = \vec{\upsilon}_0 + \vec{a}\cdot t$ (13)
where $\vec{v}_0$ and $\vec{a}$ are ***constant vectors***. By projecting expression (13) over two axes Ox, Oy one gets

$$\upsilon_x = \upsilon_{0x} + a_x t\_and\_\upsilon_y = \upsilon_{0y} + a_y t \qquad (14)$$

Then, one applies the area technique (as for 1D motion) for $\upsilon_x = \upsilon_x(t)$, $\upsilon_y = \upsilon_y(t)$ graphs and finds out that

$$x - x_0 = \upsilon_{0x}t + a_x\frac{t^2}{2} \qquad \text{and} \qquad y - y_0 = \upsilon_{0y}t + a_y\frac{t^2}{2} \qquad (15)$$

-The expressions (14) and (15) have practically the same form as in the case of 1D kinematics. So, for a particle moving ***at constant acceleration*** in a plane (*2D motion*), the following relations[2] apply:

$$\upsilon_x = \upsilon_{0x} + a_x t \qquad \upsilon_y = \upsilon_y + a_y t$$

$$x = x_0 + \upsilon_{0x}t + a_x\frac{t^2}{2} \qquad y = y_0 + \upsilon_{0y}t + a_y\frac{t^2}{2} \qquad (16)$$

$$\upsilon_x^2 = \upsilon_{0x}^2 + 2a_x(x - x_0) \qquad \upsilon_y^2 = \upsilon_{0y}^2 + 2a_y(y - y_0)$$

Notes 1] The ***motion in a plane*** can be ***decomposed into two motions*** along two directions perpendicular to each other. Each of these motions can be studied independently; the **time is the same** for both.
2] Another set of equations for Oz axe should be added in (16) if the object moves in 3D space.

[2] Both sets have the same mathematical form as those of 1-D kinematics.

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2] PROJECTILE MOTION

- One uses the term "projectile" for an object that, after being given an ***initial velocity***, moves in space ***only*** under the action of ***gravitation force***. *This is a first step approximation model that neglects* all other effects (air resistance, earth rotation …), and considers that the projectile (shot bullet, missile, golf ball,..) ***moves*** all time ***<u>in a vertical plane</u>*** **.** So, one can chose two axes Ox, Oy and apply the relations (16) to describe its motion.

- In general, one selects Oy axe along the vertical *pointing up,* Ox axe along the horizontal and the origin of coordinate system O on the ground level or at the point where the projectile starts its motion.
The relations (16) show that one can decompose the projectile 2D motion into two independent motions:
a) Horizontal one at *constant velocity* $\boldsymbol{\upsilon_x = \upsilon_{0x}}$***; there is no horizontal acceleration(***$\boldsymbol{a_x = 0}$***) to modify it.***
b)Vertical one at acceleration $\boldsymbol{a_y = -g}$***; gravity acceleration vector directed opposite to axe Oy direction.***
Remember: The gravity is the only force considered in this model. There is no motion along Oz axe because the initial velocity along Oz is zero ($\boldsymbol{\upsilon_{0z} = 0}$***) and it is not modified in time.***
Next, with $\boldsymbol{a_x = 0}$ and $\boldsymbol{a_y = -g}$ the expressions (16) become

$$\upsilon_x = \upsilon_{0x} \qquad\qquad \upsilon_y = \upsilon_{0y} - gt$$
$$x = x_0 + \upsilon_{0x}t \qquad\qquad y = y_0 + \upsilon_{0y}t - g\frac{t^2}{2} \tag{17}$$
$$\upsilon_y^2 = \upsilon_{0y}^2 - 2g(y - y_0)$$

The two following examples cover the main situations in "projectile motion problems"

Ex_1 : A ball ***dropped*** from the 6m high window of a train, moving horizontally at 50m/s, touches ground.
Find: a) Its flight time interval; b) Its horizontal displacement when touching ground;
c) Flight time if the train was at rest; d) The speed at the moment the ball touches the ground.

One starts counting the time(t = 0s) from the moment one drops the ball. As this motion happens in a plane one needs only two axes (Ox, Oy). Next, one fixes the origin of the coordinative system on the ground vertically down the drop point ; so, at t = 0, the ball coordinates are (x = 0m, y = 6 m).
So, one has to study the motion of a projectile with $\boldsymbol{x_0} = 0$, $\boldsymbol{\upsilon_{0x}} = 50$ m/s and $\boldsymbol{y_0} = 6$m , $\boldsymbol{\upsilon_{0y}} = 0$.

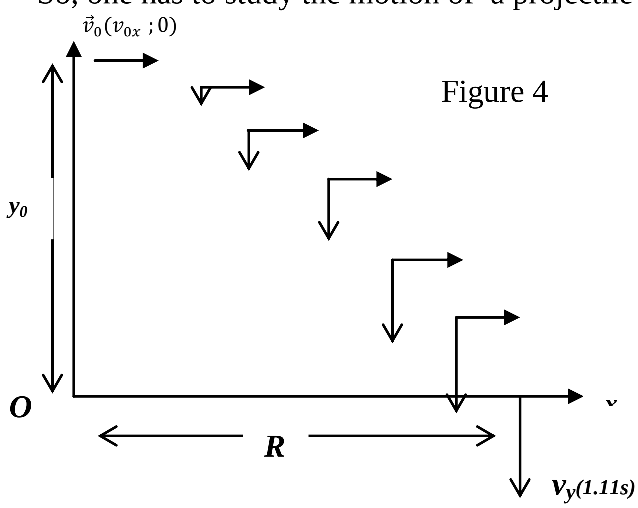

Figure 4

a) The flight time is defined by the vertical motion of projectile from y_o= 6m to y = 0m.
As $\boldsymbol{\upsilon_{0y}} = 0$, from relation (17) one get
$\boldsymbol{0 = 6m + 0*t - g*t^2/2}$ and $\boldsymbol{t^2 = 12m / 9.8m/s^2}$
From the two solutions $t_1=1.11$ *and* $t_2=-1.11s$ *only the first one has <u>physical meaning</u>.*
So the flight time of the ball is 1.11s.

b) The horizontal motion at constant velocity $\boldsymbol{v_{0x} = v_0 cos\theta = 50cos0^0 = 50m/s}$ follows for 1.11s. (as long as the ball has not touched the ground) **.** So, the horizontal shift (or range) is R = 50*1.11= 55.5m ("R stands for the range")

c) The same condition defines the flight time when the train is "at rest". So, one gets the same flight time t =1.11s, but in this case there is no x-change, i.e. R=0.
d) The *instantaneous speed* when the ball touches ground is equal to magnitude of velocity at this moment. So, one has to find two components of velocity at this moment. The horizontal component is $\upsilon_x = \upsilon_{ox} = 50$m/s.

The vertical component can be found from the last relation at (17). As $y_{fin} = 0$, $y_0 = 6$m and $\upsilon_{0y} = 0 m/s$, it comes out that $v_y^2 = 0 + 2(-g)(y - y_0) = 2gy_0 = 2 * 9.8 * 6 = 117.6$ m²/s². Next, the formula for the magnitude of a vector gives $v^2 = v_x^2 + v_y^2 = 117.6 + 2500 = 2617.6$ m²/s² and $sp = |\vec{v}| = 51.2$ m/s. Note that when touching ground its **velocity vector** is $\vec{v} = (50\hat{\imath} - 10.84\hat{\jmath})$ [m/s]

Ex_2 : A bullet is fired ***from the ground*** with initial velocity $\vec{\upsilon}_0$ at the angle θ with horizontal. Find:

a) The flight time; b) The horizontal shift to the point it touches the earth(***R-range***).
b) The shape of its path; d) The maximum height it arrives.

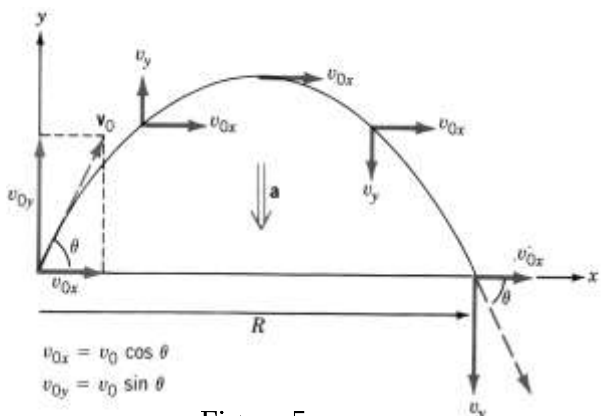

Figure 5

a) One places origin on the ground. From the figure 5, one finds out the two components of initial velocity as

$$\upsilon_{0x} = \upsilon_0 \cos\theta\_and\_\upsilon_{0y} = \upsilon_0 \sin\theta \quad (18)$$

Next, by using those velocity components at expressions (17), for $x_0 = 0$ and $y_0 = 0$ one gets

$$\upsilon_x = \upsilon_0 \cos\theta \qquad \upsilon_y = \upsilon_0 \sin\vartheta - gt$$

$$x = \upsilon_0 \cos\theta * t \qquad y = \upsilon_0 \sin\theta * t - g\frac{t^2}{2}$$

$$\upsilon_y^2 = \upsilon_{0y}^2 - 2g(y - y_0)$$

The ***flight time*** corresponds to the moment when the bullet touches the earth surface: y = 0 .

So, $$0 = \upsilon_0 \sin\theta * t - g\frac{t^2}{2} \rightarrow g\frac{t^2}{2} = \upsilon_0 \sin\theta * t \Rightarrow\Rightarrow\Rightarrow t_{flight} = \frac{2\upsilon_0 \sin\theta}{g} \quad (19)$$

b) The *range* ***R*** is defined from the flight time and horizontal velocity $\upsilon_{0x} = \upsilon_0 \cos\theta$ as follows:

$$R = \upsilon_{0x} * t_{flight} = \upsilon_0 \cos\theta * \frac{2\upsilon_0 \sin\theta}{g} = \frac{\upsilon_0^2 * 2\sin\theta\cos\theta}{g} = \frac{\upsilon_0^2 \sin 2\theta}{g} \quad (20)$$

c) The trajectory shape can be defined from the relation ***y = y(x)***. To find this expression one isolates the "time variable" at x- expression $t = \frac{x}{\upsilon_0 \cos\theta}$ and substitutes this at y-expression. So, one gets

$$y = \upsilon_0 \sin\theta * \frac{x}{\upsilon_0 \cos\theta} - \frac{g}{2}\frac{x^2}{\upsilon_0^2 \cos^2\theta} \Rightarrow\Rightarrow\Rightarrow y = \tan\theta * x - \frac{g}{2\upsilon_0^2 \cos^2\theta} * x^2 \quad (21)$$

This last expression has the form of the parabola equation ***y =Ax +Bx²***. So, the path is a parabola.

d) The maximum height corresponds to the moment when $\upsilon_y = 0$. One can find this moment from the equation

$$0 = \upsilon_0 \sin\vartheta - gt \Rightarrow\Rightarrow\Rightarrow t_{max} = \frac{\upsilon_0 \sin\theta}{g} \quad (22)$$

The height at this moment is equal to ***ymax***. So $y_{max} = \upsilon_0 \sin\theta * \frac{\upsilon_0 \sin\theta}{g} - \frac{g}{2}\left(\frac{\upsilon_0 \sin\theta}{g}\right)^2$ and

$$Max\_Height = (\upsilon_0 \sin\theta)^2\left(\frac{1}{g} - \frac{1}{2g}\right) = \frac{\upsilon_0^2 \sin^2\theta}{2g} \quad (23).$$

Homework: Do numerical calculations for $|\vec{\upsilon}_0| = 100 m/s$ and $\theta = 55^0$.

3] UNIFORM (*constant speed*) CIRCULAR MOTION OF A POINT PARTICLE

- In this type of motion the **magnitude** of the velocity vector remains **constant** all time while its **direction changes** following the tangent to the circular path. As the vector ***of velocity changes***, this is an **accelerated motion**. At first, we will find the *direction* of acceleration ***vector*** and then, its *magnitude*.

- Consider a car moving at a ***constant speed*** on a straight section of highway followed by a 90^0 right turn. The velocities $\vec{\upsilon}_1, \vec{\upsilon}_2$ have the same magnitude but different directions. If the time interval is $\Delta t = t_2 - t_1$,

the car get the average acceleration $$\vec{a}_{Aver} = \frac{\Delta \vec{\upsilon}}{\Delta t} = \frac{1}{\Delta t}(\vec{\upsilon}_2 - \vec{\upsilon}_1) \qquad (24)$$

From this expression comes out that the ***direction of acceleration*** is the ***<u>same as</u>*** that of ***vector*** $(\vec{\upsilon}_2 - \vec{\upsilon}_1)$. The figure 6.a shows that the direction of this vector is "***inside the turn and directed to turn center*** ".

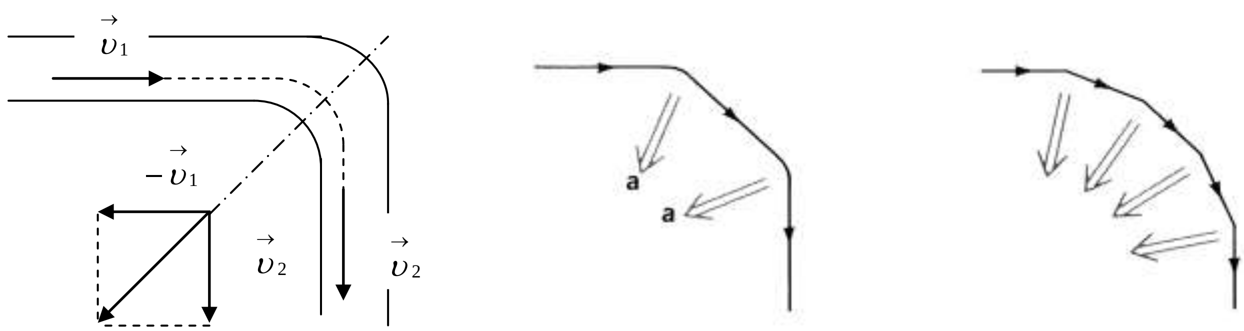

Figure 6.a 6.b 6.c

The figures 6.b,c show the situations where the turn is realized by two 45^0 and by four 22.5^0 consecutive small turns. All those acceleration vectors are directed "***inside and pass by the center of the turns"***. When the number of turns increases, the curved path fits to a portion of a circular shape and, at the limit, one deals

with the ***instantaneous acceleration*** at a point on a circular path. $$\vec{a}_c = \lim_{\Delta t \to 0} \frac{\Delta \vec{\upsilon}}{\Delta t} \qquad (25)$$

Conclusion: ***The acceleration vector is center-seeking (centripetal) at each point on a curved path.***

-Let's find its magnitude. Consider a particle moving on a circle at a **constant speed** "***υ***". During an interval

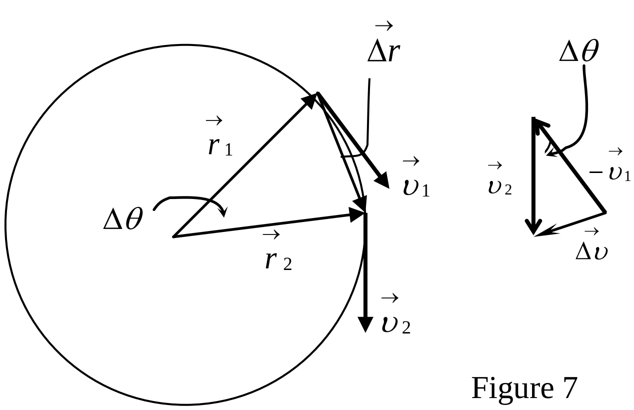

Figure 7

of time "Δt" it is shifted from position $\vec{r}_1$ to that $\vec{r}_2$. This corresponds to a small rotation by $\Delta\theta$ on circle. As the magnitudes of *position* vectors $\vec{r}_1$, $\vec{r}_2$ are equal (let's note this magnitude as "***r***") and if the arch is small, one can find the length(*magnitude*) of the *displacement vector* as

$$\left|\Delta \vec{r}\right| = \left|\vec{r}_2 - \vec{r}_1\right| = r * \Delta\theta \_ and \_ \Delta\theta = \frac{\left|\Delta \vec{r}\right|}{r} \qquad (26)$$

Being tangent to path, the velocity vectors are perpendicular to the vectors$\vec{r}_1, \vec{r}_2$. As the angle between $\vec{r}_1, \vec{r}_2$ is $\Delta\theta$, it comes out that the angle between $\vec{\upsilon}_1, \vec{\upsilon}_2$ is $\Delta\theta$, too.

From the triangle of velocities, by taking into account that velocities have the same magnitude "$\boldsymbol{\upsilon}$", one finds out that the magnitude of vector " ***velocity change*** " is $\left|\Delta \vec{\upsilon}\right| = \upsilon * \Delta\theta$ _and_ $\Delta\theta = \dfrac{\left|\Delta \vec{\upsilon}\right|}{\upsilon}$ (27)

By equalizing the equations (25,26) it comes out that $\dfrac{\left|\Delta \vec{\upsilon}\right|}{\upsilon} = \dfrac{\left|\Delta \vec{r}\right|}{r} \Rightarrow\Rightarrow \left|\Delta \vec{\upsilon}\right| = \upsilon \dfrac{\left|\Delta \vec{r}\right|}{r} = \dfrac{\upsilon}{r}\left|\Delta \vec{r}\right|$ (28)

Form (25) and (27) one finds out that $\left|\vec{a}_c\right| = \left|\lim_{\Delta t\to 0} \dfrac{\Delta \vec{\upsilon}}{\Delta t}\right| = \left|\lim_{\Delta t\to 0} \dfrac{\upsilon}{r}\dfrac{\Delta \vec{r}}{\Delta t}\right| = \dfrac{\upsilon}{r}\left|\lim_{\Delta t\to 0} \dfrac{\Delta \vec{r}}{\Delta t}\right| = \dfrac{\upsilon^2}{r}$

- So, in an ***uniform circular motion***, the acceleration is all time ***central seeking*** and its ***magnitude*** is

$$\left|\vec{a}_c\right| = \frac{\upsilon^2}{r} \qquad (29)$$

-For a point moving <u>*at constant speed on a circle*</u> one introduces the concept of **period T,** i.e. the *time it takes for a full revolution*. As the ***circumference*** of ***circle*** is **2πr** and the particle moves at constant speed on it, it comes out that the time for a full revolution is $T = \dfrac{2\pi * r}{\upsilon}$ i.e. $\upsilon = \dfrac{2\pi * r}{T}$. Then, by substituting this expression for speed at (29) one find out that

$$\left|\vec{a}_c\right| = \frac{4\pi^2}{T^2} * r \qquad (30)$$

4] NONUNIFORM (changing speed) MOTION OF A PARTICLE ON A CURVED PATH

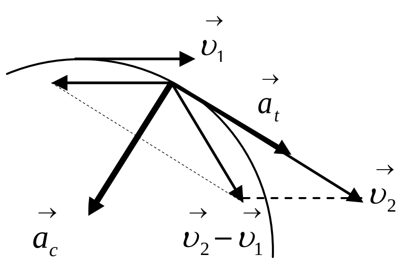

Figure 8

Consider a particle ***speeding up*** around a circle CW. Its velocity $\vec{\upsilon}_1$ at a moment t_1 becomes $\vec{\upsilon}_2$ at the next moment $t_2 = t_1 + \Delta t$. From the definition of acceleration ($\vec{a} = \frac{\Delta \vec{v}}{\Delta t}$) it comes out that its direction is the same as $\Delta \vec{\upsilon} = \vec{\upsilon}_2 - \vec{\upsilon}_1$. As $\Delta \vec{\upsilon}$ is not aligned on the radial direction, it comes out that $\vec{a}$ also is not aligned on radial direction.

So, one decomposes $\vec{a}$ in two components (see fig.8); one radial $\vec{a}_c$ and one tangential $\vec{a}_t$. The radial or ***centripetal*** acceleration is due only to the change of direction of velocity and its **magnitude** is $a_c = \dfrac{\upsilon^2}{r}$ where "$\boldsymbol{\upsilon}$" is the **instantaneous speed** and **"r"** the **radius of circle (*or path curvature at the given point*).**

The magnitude of ***tangential acceleration*** is equal to the ***derivative*** *of "* velocity *magnitude = speed "*

$$a_t = \frac{d|\vec{v}|}{dt} \qquad (31)$$

When **speeding up** this derivative is **positive;** $\vec{a}_t$, $\vec{\upsilon}$ vectors have the **same direction** in space. When the particle is **slowing down**, this derivative is **negative** and the vector $\vec{a}_t$ has **opposite direction** versus $\vec{\upsilon}$.