

LECTURE 9

1] INTRODUCTION

- The **work** and the **energy** are two main physics parameters that *simplify* the **study** of motion of objects. Especially, if the object moves on a *curved path* and the *forces exerted on it change* (magnitude, direction or both) during its motion (like *spring's force magnitude or friction force direction along a curved surface.....*), one prefers to use these parameters instead of Newton's laws.

- When lifting a body up in uniform way (zero acceleration) one exerts a force " F_{up} " with equal magnitude to that of body weight " F_G ". The everyday experience shows that:

- For the **same** vertical **shift**, **larger the weight** is, more **physical effort** is needed;
- For the **same weight**, **bigger the vertical shift** is, more **physical effort** is needed;

So, one **notes** that the product "**force * shift**" can be used to **measure** of the *exerted effort when moving up an object*. This intuitive information opens the way for the precise definition of **mechanical work**. Similarly, one may figure out the **energy** as the "*capacity of a source to provide mechanical work*". From those two intuitive points of view one can get to the conclusion that "**the work provided by a source should be equal to the change of energy amount contained in the source that provides the work**". Note that this point of view is true as long as one deals only with pure mechanics phenomena without production of heat or other types of energy.

2] THE WORK DONE BY A CONSTANT FORCE

- One says that a **constant force** \vec{F} has done work **only if the object** where it is applied gets **shifted**.

If the object **displacement** is \vec{s} then, the force has achieved the work $W = \vec{F} * \vec{s} = F * s * \cos\theta$ (1)

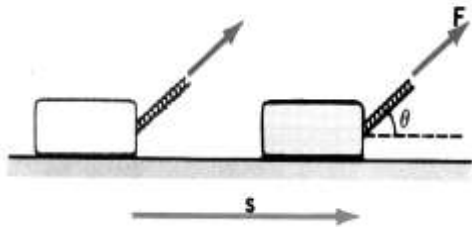


Figure 1

" θ " is the angle between the *direction of force* and the *direction of displacement vector*. Note that :

- Work is provided by **the source** that exerts the force.
- If the **displacement is zero**, there is **zero work** even if the source makes an effort (*applies the force*).
- Only the *component of force along the displacement direction achieves work*. The *component of force perpendicular to displacement achieves zero work* because in this case $\theta = 90^0$ and $\cos 90^0 = 0$.
- Work provided by a force is negative if $\theta > 90^0$.

- As one has defined it by a scalar product, the **work is a scalar physical** quantity. Its unit is a *derived unit*; its unit in SI system is the Joule (J);

$$1J = 1N * 1m \quad (\text{force applied in displacement direction}) \quad (2)$$

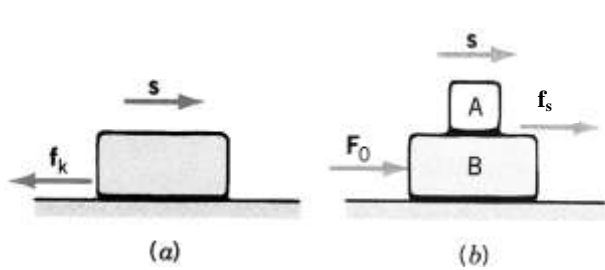
- One may calculate the work done by a constant force \vec{F} during a displacement \vec{s} by using the components of those two vectors in any Cartesian frame Oxyz by using the dot product expression

$$W = \vec{F} * \vec{s} = F_x * s_x + F_y * s_y + F_z * s_z = F_x * \Delta x + F_y * \Delta y + F_z * \Delta z \quad (3)$$

Remember a) $\vec{F} = F_x \vec{i} + F_y \vec{j} + F_z \vec{k}$ and $\vec{s} = \Delta x \vec{i} + \Delta y \vec{j} + \Delta z \vec{k}$ b) Rules of scalar product

Note that the numerical value of **work** does not depend on the reference frame, because the magnitudes of vectors \vec{F} , \vec{s} and the angle between them (see expression 1) do not depend on the selected frame.

- When a block slides over a *horizontal plane*, the work done by \vec{F}_G or \vec{N} is zero because they are both perpendicular to displacement vector. For the same reason the work by centripetal force is always zero. In the case of the **work done by friction** force, before starting the calculations, one must verify carefully the *directions of displacement and friction force vectors*. In the case shown in fig 2.a, the friction force on the block has opposite direction versus displacement vector and the friction achieves a negative work



$$W_f = \vec{f}_k * \vec{s} = f_k * s * \cos 180^\circ = -f_k * s \quad (4)$$

on the block; the **source** of friction force is the floor.

In the case of **fig 2.b**, the force F_0 pushes the block **B** right-side and the block B pulls on block A to the

Figure 2

right via the static friction force between them. Due to its *inertia*, the block A tends to keep its motion status and this way it would tend to slide left-side over block B. As the **static friction** force on the block A has the **same direction** as its **displacement**, the **work by friction on block A is positive**; $W_f = +f_s * s$

-When a set of forces $\vec{F}_1, \vec{F}_2, \dots, \vec{F}_n$ is acting **on the same body** that is shifted by \vec{s} , each force achieves work independently of the others.

$$W_1 = \vec{F}_1 * \vec{s}; \quad W_2 = \vec{F}_2 * \vec{s}; \quad \dots \quad W_n = \vec{F}_n * \vec{s} \quad (5)$$

Total work achieved on the body is $W_{NET} = W_1 + W_2 + \dots + W_n = (\vec{F}_1 + \vec{F}_2 + \dots + \vec{F}_n) * \vec{s} = \vec{F}_{NET} * \vec{s}$ (6)

$$\vec{F}_{NET} = \sum_i^n \vec{F}_i \text{ is the vector sum of all forces exerted on the body.}$$

- As mentioned previously, the **work** done by a **force** can be **positive** (if $\theta < 90^\circ$) or **negative** (if $\theta > 90^\circ$).

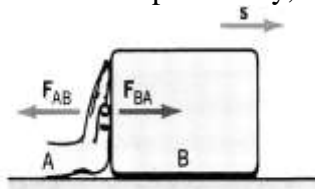


Figure 3

In figure 3, the hand "object A" pushes the block "object B" and shifts it by the **displacement** \vec{s} . This way, via the force exerted on B, the source of this force i.e. "object A" **achieves the positive work**

$$W_{on_B_by_A} = \vec{F}_{BA} * \vec{s} = F_{BA} * s * \cos 0^\circ = F_{BA} * s \quad (7)$$

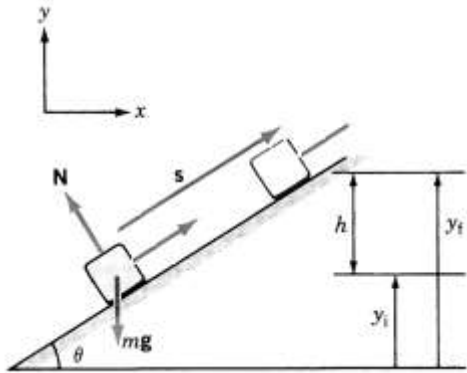
The third law of Newton tells that the block "object B" exerts on the hand "or object A" a force with equal magnitude but opposite direction to \vec{F}_{BA} . $\vec{F}_{AB} = -\vec{F}_{BA}$ (8)

The work achieved by the block B on the hand (\vec{F}_{AB} work), after the hand is shifted by \vec{s} , is

$$W_{on_A_by_B} = \vec{F}_{AB} * \vec{s} = -\vec{F}_{BA} * \vec{s} = -W_{by_A_on_B} \quad (9)$$

So, when the object A (hand) achieves a positive work on the object B (block), the object B (block) achieves the same amount of work (but with negative sign) on the object A.

- The work done by gravity force is an important step for introduction of energy concept. Consider a block that slides on an inclined plane due to a pulling force (fig.4). Let's ignore the action of other forces and **concentrate on the work done by weight force**. The set of axes selected in fig. 4 is not good to study the motion by Newton law¹ but it allows inferring some important results and *simplifies the calculation of work* by gravity. In this Oxy frame, the components of gravity force are $\vec{F}_G(0, -mg)$ and those of the displacement are $\vec{s}(\Delta x, \Delta y)$. By applying the formula (3), one gets



$$W_G = (0\hat{i} - mg\hat{j}) * (\Delta x\hat{i} + \Delta y\hat{j}) = -mg\Delta y$$

$$W_G = -mg(y_f - y_i) \tag{10}$$

If the block has moved upside, $y_f > y_i$ and the **work done by gravity force** on the block is **negative**.

If the block **slides down**, $y_f < y_i$ and the work done by gravity force on the block is **positive**.

Figure 4

Note: If, initially, the block is moved up (ex. by hand) from height y_i to y_f and next left free to slide down to y_i , the **total work** done by gravity force on the block is **zero**

$$W_{tot} = W_G^{up} + W_G^{down} = [-mg(y_f - y_i)] + [-mg(y_i - y_f)]$$

$$W_{G-tot} = -mg * [(y_f - y_i) + (y_i - y_f)] = 0.$$

So: The **work** provided by the **gravity** depends *only on* the **initial and final heights** of object and **not from the path** followed during its motion. This means that, **the work done by gravity is zero for any closed path** (final position of object is the same as its initial position).

3] THE AREA TECHNIQUE FOR WORK CALCULATIONS IN ONE DIMENSIONAL SPACE

- For a **constant force in one dimension space** $\vec{F} = F\hat{i}$; $\vec{\Delta s} = \Delta x\hat{i}$ and $W = F\Delta x = F(x_f - x_i)$ (11)

This **work is equal to the shaded area under the graph** $F = F(x)$ shown in figure 5.a. One may set the origin of Ox at x_i , note $x_f = x$ and the work $W = F * x$ corresponds to the area counted from origin.

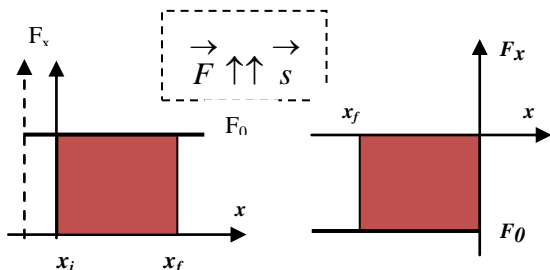


Figure 5.a ($W > 0$)

Figure 5.b ($W > 0$)

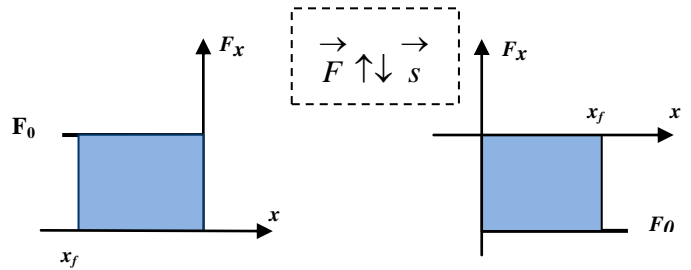


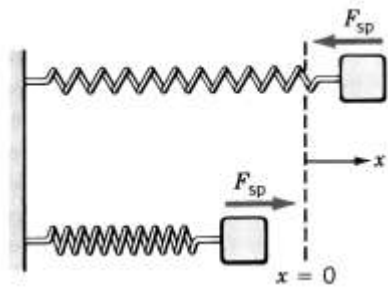
Figure 5.c ($W < 0$)

Figure 5.d ($W < 0$)

Note that when this area is located in **first or third** quadrant (fig.5.a, b) it is positive i.e. one deals with a **positive work**; when it is located in the **second or fourth** quadrant (fig.5.c, d) it shows a **negative work**.

¹ To use Newton's laws one would select Ox axe along the motion direction (i.e. same direction as displacement vector)

- The *area technique* offers an suitable way to calculate the work done by a 1D **variable force** , i.e. the work done by " a *different applied force* at different *locations* of object" . The **elastic force of a spring on a block** is a good example for such type of forces; when compressed or extended, an ideal (*massless*) spring, exerts at its moving end (see figure 6) the "*elastic or restoring force*" given by **Hook's law**

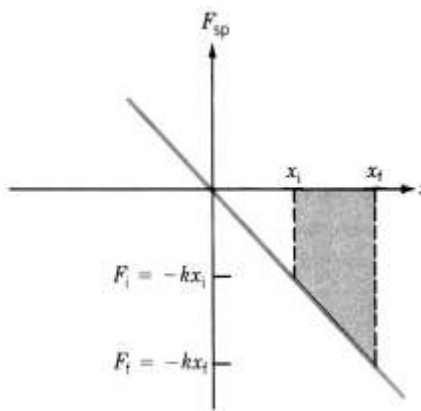


$$\text{(Hook's law)} \quad F_{sp} = -k * x \quad (12)$$

k [N/m] is the spring constant; x is the **displacement of moving end** from equilibrium.

This expression assumes that $x_i = 0$ (equilibrium) and this way the **displacement** " Δx " is equal to the **algebraic value** of x . When $x > 0$ (spring **extended**) the force $F_{sp} < 0$ and when $x < 0$ (spring **compressed**) the force $F_{sp} > 0$. So, in any case, the **spring's force tents to restore the equilibrium** position of spring. **The forces that tent to bring the system to its equilibrium configuration are known as restoring forces.**

Figure 6



- The figure 7 shows the graph of expression (12). The **work done** by an **elastic force** when its application point (**spring end**) is shifted from x_i to x_f is given by the shaded area under the graph. One can calculating this area as the difference of the two triangles' areas and find out that the work done by the spring is

$$W_{sp} = (-kx_f * \frac{x_f}{2}) - (-kx_i * \frac{x_i}{2}) = \frac{k}{2}(x_i^2 - x_f^2) = -\frac{k}{2}(x_f^2 - x_i^2) \quad (13)$$

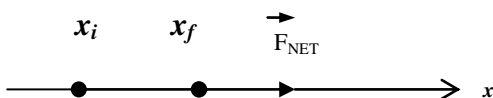
-The expression (13) shows that the **work done by a spring force (elastic force) depends only on initial and final positions** of its moving end (or **tail of force**). Spring's **work** is **positive** when the **direction of its force** is the **same** as that of **displacement**. This happens when the block is **moving versus equilibrium** point, i.e. when $|x_i| > |x_f|$. The spring's work is **negative** when the **direction of restoring force** is **opposite** to that of **displacement**. This happens when the block is **moving away from equilibrium** point, i.e. when $|x_i| < |x_f|$.

Also, **for a full oscillation, the work done by spring is zero** because $x_f = x_i$.

Note: The area technique may be used for any variable force; keep in mind that the *sign of work* depends on *sign of area under the graph* and the *position of initial point* versus the *position of final point*, too.

3] THE THEOREM WORK – ENERGY (ONE DIMENSION SPACE)

-Consider a **particle** (object) with mass " m " in translational motion with acceleration " a " along the direction Ox under the effect of a **net constant force** F_{NET} directed along the same direction(see fig.8). The work achieved by the force F_{NET} during a displacement of the particle $\Delta x = x_f - x_i$, is



$$W_{Net} = F_{NET} * \Delta x * \cos 0^\circ = (ma) * \Delta x = m * (a\Delta x) \quad (14)$$

Figure 8

By using the kinematics' relation between initial and final velocities one gets

$$v_f^2 - v_i^2 = 2a\Delta x \Rightarrow a\Delta x = \frac{1}{2}v_f^2 - \frac{1}{2}v_i^2 \quad (15)$$

and by substituting in eq. (14)
$$W_{NET} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 \quad (16)$$

- Definition: The quantity
$$K = \frac{1}{2}mv^2 \quad (17)$$
 is a parameter known as the **KINETIC ENERGY** of the particle.

The **net work** done by the net force F_{NET} on particle is equal to the **difference of particle K-values**

$$W_{NET} = \Delta K = K_f - K_i \quad (18)$$

The relation (18) is the mathematical expression of the **Work-Energy theorem** which states that:

The net work done on a particle is equal to the change of its kinetic energy.

- The expression (17) shows that the numerical value of kinetic energy function depends on the frame used to calculate the particle velocity. In another **inertial frame** (in uniform motion versus a first one), the velocity of the same particle is different and its kinetic energy has another value. This comparison helps understand that the **numerical value of kinetic energy** by itself is **not very meaningful**. However, the **change of kinetic energy is independent on the reference frame** and it does have a **precise physical meaning**; it is equal to the net work achieved on the particle. So, the **energy** is introduced in physics as a **mathematical parameter** which **change** is equal to **work**.

-For problems related to everyday experience, one assumes that velocity is calculated versus an inertial frame tied to the earth and one finds out that the **kinetic energy** of a terrestrial object is the **work needed to increase its speed from zero to a given value v** (put $K_i = 0$ in eq. (18) and get $W_{NET} = K_f = \frac{1}{2}mv^2$.

Equally, one may refer to the situation when the object velocity is decreased from v to **zero** due to the stopping action from outside. In this case, $K_f = 0$ and the **net work done on the object is negative**

$$W_{NET} = -K_i = -\frac{1}{2}mv_i^2$$

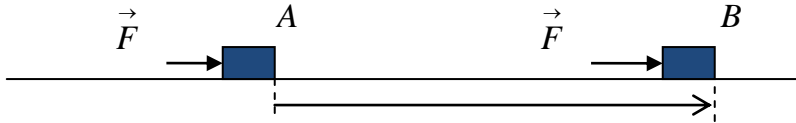
Remembering the third law, one infers that in this case, the considered object achieves **positive work** on the surrounding medium and by doing this it spends its kinetic energy. Note that this assertion fits with the idea about the energy as the capacity of this object to provide work.

Important note: We derived the **work-energy theorem** for the case of a **constant force** acting in a **one-dimensional** space. This **theorem** remains **valid** even for **forces**

- that depend on position (like spring force),
- that act in a **two or three-dimensional space**.

4] THE POWER

- Suppose that one needs to shift an object from point A to point B. This action requires to provide a certain amount of work and one can *choose between different sources* that can provide this work. Which one to chose? If one decides to use "**the source that achieves faster displacement**" one has to consider another physical parameter: "**the mechanical power**".



-The **mechanical power** is the **rate** at which a **given work** is done. If the "source" delivers an amount of work ΔW during the interval of time Δt , one says that it is delivering **the average power**

$$P_{AV} = \frac{\Delta W}{\Delta t} \quad (19)$$

The *unit* of power in SI system is the "**Watt**"

$$1W = 1J/1s \quad (20)$$

Note: In car industry, one uses widely another unit of power "horsepower"; **1hp = 745.669W**

One defines the **instantaneous mechanical power** as

$$P = \lim_{\Delta t \rightarrow 0} \frac{\Delta W}{\Delta t} \equiv \frac{dW}{dt} \quad (21)$$

- One may consider that, during the *infinitesimal displacement* \vec{ds} , the "source" is acting on the object by a *constant force* \vec{F} . This means that the source provides the work $dW = \vec{F} * \vec{ds}$ (22)

If this displacement happens during the *infinitesimal* the time interval dt , then *the object is moving*

at *instantaneous velocity* $\vec{v} = \frac{d\vec{s}}{dt}$ from which one derives $d\vec{s} = \vec{v} dt$ (23)

After substitution of relation (23) in (22) one gets

$$dW = \vec{F} \vec{v} dt \quad (24)$$

and finally (based on (21)) the *instantaneous power*

$$P = \frac{dW}{dt} = \vec{F} * \vec{v} \quad (25)$$

The expression (25) is based on **instantaneous vectors** (force and velocity). Note that the *power is a scalar physical parameter* that may *change in time*. But, if the force and velocity are constant vectors, the power is constant, too. In this case, one says that the *source provides a constant power in time*.

- From a formal point of view, the **work done by the source** of the force on an object can be seen as *the portion of mechanical energy transferred* from the "source to the adjacent regions of space". This way, the **delivered power [watt]** from a "source" would be expressed as

$$P = - \frac{dE}{dt} \quad (26)$$

where E [Joules] stands for the **total mechanical energy** of the "source" that provides the **mechanical power**. The sign "-" in expression (26) is related to the fact that the *energy of the source is decreased when it provides energy* ($E_f < E_{in}$ and $E_f - E_{in} < 0$) while the provide power is **positive** in the sense that it **does positive work** on the object that receives it.