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LECTURE 9

1] INTRODUCTION

- The ***work*** and the ***energy*** are two main parameters that ***simplify*** the ***study*** *of objects motion.* Especially, if the object moves on a *curved path* and the *forces* exerted *on it change (magnitude, direction or both)* during its motion *(like magnitude of spring's force or direction of friction force)*, one has found that those two parameters offer easier solutions methods than " $m, \vec{a}, \vec{F}$ " and direct application of Newton's laws.

- When lifting a body up in a uniform way (*zero acceleration*) one exerts a force with *magnitude* "F_{up} " equal to that of the body *weight* " F_G". The everyday experience shows that:

- For the **same** vertical **shift**, ***larger*** *the* ***weight*** *is, more* ***physical effort*** is needed;
- For the **same weight**, ***bigger*** *the vertical* ***shift*** *is, more* ***physical effort*** is needed;

So, one ***can refer to*** the product " ***force**** ***shift*** " to **measure** the *exerted effort.* This intuitive information opens the way for the precise definition of ***mechanical work***. In a similar way, one may figure out the ***energy*** as the " *capacity of a source to provide mechanical work* ".
Those two intuitive points of view guide to the conclusion that " *the* ***work*** *provided by a source should be* ***equal to the change of energy*** *amount* ***contained in the source*** *that provides that work* ".
Note that this point of view is true as long as one deals only with pure mechanics phenomena *without production of heat or other types of energy*.

2] THE *WORK* PROVIDED *BY* A *CONSTANT FORCE*

- One says that a ***force*** has done work ***only if its tail*** (and ***object*** on which it is applied) gets ***shifted***. If its tail ***displacement*** is $\vec{s}$, the force $\vec{F}$ *has achieved the work* $W = \vec{F} * \vec{s} = F * s * \cos\theta$ (1)

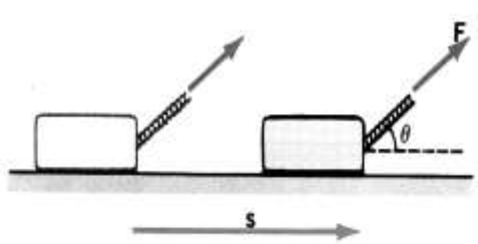


Figure 1

" θ " is the angle between the *direction of force and the direction of displacement vector*. Note that :

- Work is provided by ***the source*** that exerts the force.
- If the ***displacement is zero***, there is ***zero work*** even if the source makes an effort (*applies the force*).
- Only the *component of force along the displacement* direction *achieves work*. The *component of* force *perpendicular to displacement* achieves ***zero work*** because in this case $\theta = 90^0$ and *cosine*(90^0) = 0.
- Work provided by a force is negative if $\theta > 90^0$.

- Due to its definition by a scalar product, the physical parameter ***work*** *is a* ***scalar*** quantity and its unit is a *derived unit*. Its unit in SI system is the Joule (J); ***1J = 1N*1m*** (2)

- One may calculate the work done by a *constant force* $\vec{F}$ during a displacement $\vec{s}$ via the components of those two vectors in any Cartesian frame Oxyz by using the dot product expression

$$W = \vec{F} * \vec{s} = F_x * s_x + F_y * s_y + F_z * s_z = F_x * \Delta x + F_y * \Delta y + F_z * \Delta z \quad (3)$$

Remember a) $\vec{F} = F_x \vec{i} + F_y \vec{j} + F_z \vec{k}$ and $\vec{s} = \Delta x \vec{i} + \Delta y \vec{j} + \Delta z \vec{k}$ b) Rules of scalar product

Note that the numerical value of ***work*** does ***not depend*** on the reference frame, because the magnitudes of vectors $\vec{F}$, $\vec{s}$ and the angle between them (see expression 1) do not depend on the selected frame.

- When a block slides over a *horizontal plane,* the work done by $\vec{F}_G$ or $\vec{N}$ is zero because they are both perpendicular *to displacement vector*. For the same reason the ***work by centripetal force is always zero***. In the case of the ***work done by friction*** force, before starting the calculations, one must verify carefully the *directions of displacement* and *friction force vectors*. In the case shown in *fig 2.a* , the friction force on the block has opposite direction versus displacement vector and the friction achieves a negative work

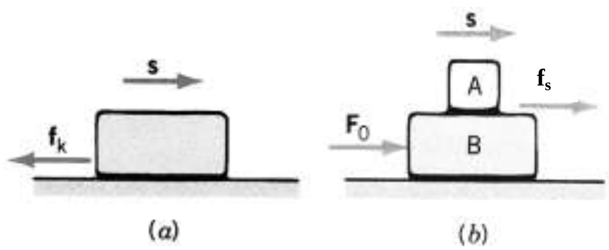


Figure 2

$$W_f = \vec{f}_k * \vec{s} = f_k * s * \cos 180^0 = -f_k * s \qquad (4)$$

on the block; the ***source*** of *friction force* is the floor.

In the case of ***fig 2.b***, the force F_o pushes the block ***B*** right-side and the block B pulls on block A to the right via the static friction force between them.

Due to *its inertia, the block A tents to* keep its motion status and tents to slide left-side over block B. So, the *static* ***friction*** force *on the block A* has the ***same direction*** as its ***displacement*** and the ***work by friction on block A is positive;*** $W_f = + f_s * s$

-When a set of forces $\vec{F}_1, \vec{F}_2, ..., \vec{F}_n$ is acting ***on the same body*** that is shifted by $\vec{s}$, each force achieves work independently of the others. $W_1 = \vec{F}_1 * \vec{s}; \_W_2 = \vec{F}_2 * \vec{s}; \_ .... W_n = \vec{F}_n * \vec{s}$ (5)

Total work *achieved* ***on the body*** is $W_{NET} = W_1 + W_2 + ..W_n = (\vec{F}_1 + \vec{F}_1 + ...\vec{F}_n) * \vec{s} = \vec{F}_{NET} * \vec{s}$ (6)

$\vec{F}_{NET} = \sum_i^n \vec{F}_i$ is the vector sum of all forces ***exerted on the body***.

- As mentioned previously, the ***work*** done by a ***force*** can be **positive** (if θ < ***90º***) or ***negative*** (if ***θ >90º***).

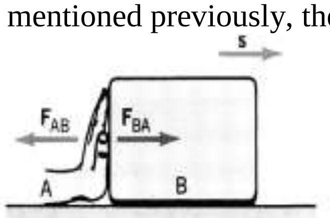


Figure 3

In figure 3, the hand "*object A"* pushes the block "*object B*" and shifts it by the **displacement** $\vec{s}$. This way, via the force $\vec{F}_{BA}$ exerted on B, the source of this force i.e. " object A" ***achieves the positive work***

$$W_{on\_B\_by\_A} = \vec{F}_{BA} * \vec{s} = F_{BA} * s * \cos 0^0 = F_{BA} * s \qquad (7)$$

The third law of Newton tells that the block "*object B"* exerts on the hand "or *object A* " a force with equal magnitude but opposite direction to $\vec{F}_{BA}$. $\vec{F}_{AB} = -\vec{F}_{BA}$ (8)

The work achieved by the block B on the hand ($\vec{F}_{AB}$ work) , after the hand is shifted by $\vec{s}$, is

$$W_{on\_A\_by\_B} = \vec{F}_{AB} * \vec{s} = -\vec{F}_{BA} * \vec{s} = -W_{by\_A\_on\_B} \qquad (9)$$

So, *when the object A (hand)* ***achieves a positive work*** *"W" on the object B (block), the object B (block) achieves the* ***same amount of work (but with negative sign*** *"-W") on the object A .*

- The work done by force of gravity is an important step for introduction of energy concept. Consider a block that slides up an inclined plane due to a pulling force (fig.4). Let's ignore the action of other forces and ***concentrate on the work done by weight force***. The set of axes selected in fig. 4 is not good to study the motion by Newton law[1] but it allows inferring some important results and *simplifies the calculation of work* by gravity. In this Oxy frame, the components of gravity force are $\vec{F}_G, (0, -mg)$ and those of the displacement are $\vec{s}(\Delta x, \Delta y)$. By applying the formula (3), one gets

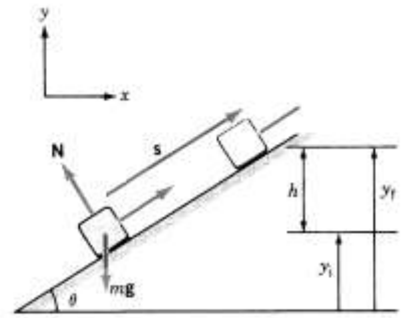


Figure 4

$$W_G = (0\hat{\imath} - mg\hat{\jmath}) * (\Delta x\hat{\imath} + \Delta y\hat{\jmath}) = -mg\Delta y$$

$$W_G = -mg(y_f - y_i) \qquad (10)$$

As the block is shifted upside, $y_f > y_i$ and the **work** done **by gravity** force on the block is ***negative***.
If the block ***slides down,*** $y_f < y_i$ and the work done by gravity force on the block is ***positive***.

Note: If, initially, the block is moved up (ex. *by hand*) from height y_i to y_f and next left free to slide down to y_i, the ***total*** **work** done *by gravity* force on the block is ***zero***

$$W_{G-tot} = W_G^{up} + W_G^{down} = [-mg(y_f - y_i)] + [-mg(y_i - y_f)]$$
$$W_{G-tot} = -mg[(y_f - y_i)] + [(y_i - y_f)] = 0$$

So: ***The work done by gravity is zero for a closed path*** *(final position of object is the same as its initial position).* In more general terms, this means that the ***work*** provided ***by*** the ***gravity*** depends *only on* the **initial and final locations** of object and ***not from the shape of path*** followed during its motion.

3] THE AREA TECHNIQUE FOR WORK CALCULATIONS IN ***ONE DIMENSIONNAL*** SPACE

- For a ***constant force in*** *one dimension space*

$$\vec{F} = F\hat{\imath};\ \overrightarrow{\Delta s} = \Delta x\hat{\imath} \quad \text{and} \quad W = F\Delta x = F(x_f - x_i) \qquad (11)$$

This ***work is equal to the shaded area under the graph F = F(x)*** shown in figure 5.a. One may set the origin of Ox at x_i, *note* $x_f = x$ and the work $W = F * x$ corresponds to the area counted from origin.

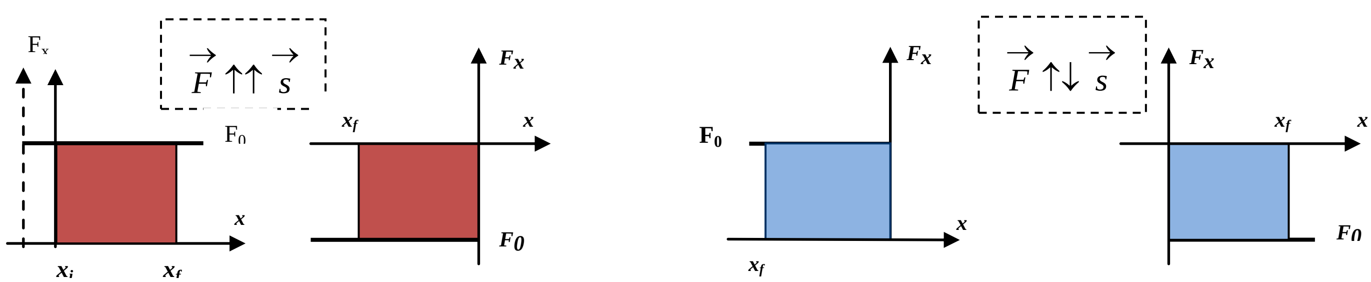


Figure 5.a (*W > 0*) Figure 5.b (W > 0) Figure 5.c (W < 0) Figure 5.d (W < 0)

Note that if *this area* is located in ***first*** or ***third*** quadrant (fig.5.a, b) it is ***positive*** i.e. one deals with a ***positive work****;* if it is located in the ***second*** or ***fourth*** quadrant (fig.5.c,d) one deals with a ***negative work***.

[1] To use Newton's laws one would select Ox axe along the motion direction (i.e. same direction as displacement vector)

- The *area technique* offers an suitable way to calculate the work done by a 1D ***variable force*** , i.e. the work done by " *a different* applied *force* at different *locations* of object" . The ***elastic force*** *of a spring* ***on a block*** is a good example for such type of forces; when compressed or extended, an ideal (***massless***) spring, exerts at its moving end (see figure 6) the "*elastic or restoring" spring force given by Hook's law*

$$F_{sp} = -k * x \quad \text{(Hook's law)} \qquad (12)$$

k [N/m] is the spring constant; ***x*** is the ***displacement of moving end*** from equilibrium.

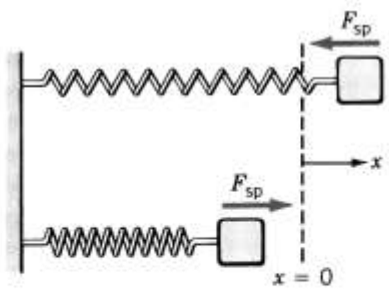


Figure 6

This expression assumes that $x_i = 0$ (equilibrium) and this makes that the ***displacement*** *"Δx" is* equal to the *algebraic value* of ***x***. When $x > 0$ (spring **extended**) the force $F_{sp} < 0$ and when $x < 0$ (spring **compressed**) the force $F_{sp} > 0$. So, in any case, the **spring's force tents to restore the equilibrium** position of spring. ***The forces that tent to bring the system to its equilibrium configuration are known as restoring forces.***

- The figure 7 shows the graph of expression (12). The ***work done*** by an **elastic force** when its application point (***spring end***) is shifted from x_i to x_f is given by the ***shaded area under the graph***. One can calculate this area as the difference of the two triangles' areas is $W_{sp} = Area(x_f) - Area(x_i)$ and find out that the work done by the spring

$$W_{sp} = \left(-kx_f * \frac{x_f}{2}\right) - \left(-kx_i * \frac{x_i}{2}\right) = \frac{k}{2}\left(x_i^2 - x_f^2\right) = -\frac{k}{2}\left(x_f^2 - x_i^2\right) \qquad (13)$$

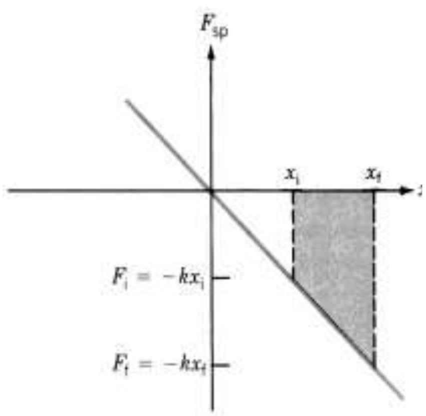


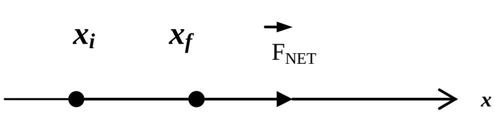
Figure 7

-The expression (13) shows that the *work done by a spring force* (**elastic force**) ***depends only on initial and final positions*** *of its moving end (or* ***tail*** *of force)*. Spring's *work* is ***positive*** when the *direction of its force* is the *same* as that of *displacement*. This happens when the block is **moving versus equilibrium** point, i.e. when $|x_i| > |x_f|$. The spring's work is ***negative*** when the direction of *restoring force is opposite* to that *of displacement* .This happens when the block is **moving away from equilibrium** point, i.e. when $|x_i| < |x_f|$.
Also, ***for a full oscillation, the work done by spring is zero*** because $x_f = x_i$.

Note: The area technique may be used for any variable force; keep in mind that the *sign of work* depends on *sign of area under the graph* *and* the *position* of *initial point* versus the *position of final* point, too.

3] THE THEOREM WORK – ENERGY (*ONE DIMENSION* SPACE)

-Consider a **particle** (object) with mass ***"m"*** in translational motion with acceleration ***"a"*** along the direction ***Ox*** under the effect of a *net* ***constant*** *force* $\vec{F}_{Net}$ directed along the same direction(see fig.8). The work achieved by the force $\vec{F}_{Net}$ during a displacement of the particle $\Delta x = x_f - x_i$, is



$$W_{Net} = F_{NET} * \Delta x * cos0^0 = (ma) * \Delta x = m * (a\Delta x) \qquad (14)$$

Figure 8

By using the kinematics' relation between initial and final velocities one gets

$$\upsilon_f^2 - \upsilon_i^2 = 2a\Delta x \Rightarrow\Rightarrow a\Delta x = \frac{1}{2}\upsilon_f^2 - \frac{1}{2}\upsilon_i^2 \quad (15)$$

and by substituting in eq. (14)

$$W_{NET} = \frac{1}{2}m\upsilon_f^2 - \frac{1}{2}m\upsilon_i^2 \quad (16)$$

- <u>*Definition*</u>: The quantity $K = \frac{1}{2}m\upsilon^2$ (17)

is a parameter labelled as the ***KINETIC ENERGY** of the particle**.*

*The **net work** done by the **<u>net force</u> F_{NET}** on **particle** is equal to the **difference of particle K-values***

$$W_{Net} = \Delta K = K_f - K_i \quad (18)$$

The relation (18) is the mathematical expression of the **Work-Energy theorem** which states that:

The net work done on a particle is equal to the change of its kinetic energy.

- The expression (17) shows that the numerical value *of <u>kinetic energy function depends on the frame used to calculate the particle velocity</u>*. In another ***inertial frame*** (*in uniform motion versus a first one*), the velocity of the same particle is different and its kinetic energy has another value. This comparison helps understand that the ***numerical value of kinetic energy*** by itself is **<u>not very meaningful</u>**. However, the **change of kinetic energy is independent on the inertial frame** where it is calculated and **it does** have a **precise physical meaning;** it is **equal** to the **<u>net work achieved on</u>** the particle**.**
In physics, *the energy* is introduced as a *mathematical parameter* which *change* is equal to *work*.

-For problems related to everyday experience, one assumes that velocity is calculated versus an inertial frame tied to the earth and one finds out that the ***kinetic energy*** of a terrestrial object is the ***work needed to increase its speed from zero to a given value υ*** (put ***K_i = 0*** in eq. (18) and get $W_{NET} = K_f = \frac{1}{2}m\upsilon^2$**.**

Equally, one may refer to the situation when the object speed is decreased from ***υ*** to ***zero*** due to the stopping action of applied forces. In this case, *K_f **= 0** and the **net work <u>done on the object</u> is negative***

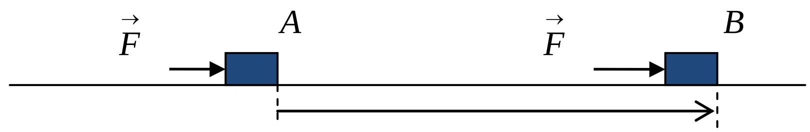
$$W_{NET} = -K_i = -\frac{1}{2}m\upsilon_i^2$$

***Remembering the third law,** one infers that in this case**,** the considered object achieves **positive work** on the surrounding medium and by doing this it spends its kinetic energy*. Note that this assertion fits well with the idea about the energy as the capacity of this object to provide work.

Important note: We derived the ***work-energy theorem*** for the case of a ***constant force*** acting in a ***one-dimensional*** space but one may prove that this ***<u>theorem</u>*** *remains **<u>valid</u>** even for **forces:***
- that depend on position *(like spring force),*
- that *act in a **two** or **three-dimensional space**.*

4] THE POWER

- Suppose that one needs to shift an object from point A to point B. This action requires to provide a certain amount of work and one can *choose between different sources* that can provide this work. Which one to chose? If one decides to use " ***the source that achieves faster displacement "*** one has to consider another physical parameter: " ***the mechanical power "***.



-*The mechanical* ***power*** *is the* **rate** *at which a* **given work** *is done*. If the "*source"* delivers an amount of work ***ΔW*** during the interval of time ***Δt***, one says that it is delivering ***the average power***

$$P_{AV} = \frac{\Delta W}{\Delta t} \tag{19}$$

The *unit* of power in SI system is the " **Watt** " ***1W = 1J /1s*** (20)

Note: *In car industry, one uses widely another unit of power " horsepower* "; ***1hp = 745.669W***

One defines the ***instantaneous power*** as $P = \lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} \equiv \frac{dW}{dt}$ (21)

- One may consider that, during the *infinitesimal displacement* $\vec{ds}$, the "*source"* is applying on the object a *constant force* $\vec{F}$. This means that the source provides the work $dW = \vec{F} * \vec{ds}$ (22)

If this displacement happens during the *infinitesimal* time interval **dt,** then *the object is moving at instantaneous velocity* $\vec{\upsilon} = \frac{d\vec{s}}{dt}$ from which one derives $d\vec{s} = \vec{\upsilon}\, dt$ (23)

After substitution of relation (23) in (22) one gets $dW = \vec{F}\vec{\upsilon}\, dt$ (24)

and finally(21) one finds out that the *instantaneous power* $P = \frac{dW}{dt} = \vec{F} * \vec{\upsilon}$ (25)

The expression (25) is based on ***<u>instantaneous vectors</u>*** (*force and velocity*). Note that the *power is a scalar physical parameter* that may *change in time*. But, if the force and velocity are constant vectors, the power is constant, too. In this case, one says that the *source provides a constant power in time*.

- From a formal point of view, the ***work done by the source*** of the force on an object can be seen as *the portion of* ***mechanical energy transferred*** *from the "source to the adjacent regions of space"*. This way, the ***delivered power*** [***watt***] from a "source" would be expressed as

$$P = -\frac{dE}{dt} \tag{26}$$

where ***E*** [*Joules*] stands for the ***total mechanical energy*** of the "***source***" that provides the ***mechanical power*** . The sign " - " in expression (26) is related to the fact that the *energy of the source is <u>decreased when it provides energy</u> ($E_f < E_{in}$ and $E_f - Ei_n < 0$) while the provided power is* ***positive*** *in the sense that it* ***does positive work*** *on the object that receives it* .