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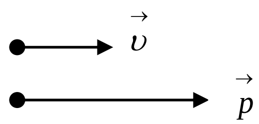
LECTURE_11

1] LINEAR MOMENTUM

- The study of ***collisions between*** different ***objects*** showed that one can explain the experimental results by referring to the changes of a particular ***physical parameter*** " the ***vector of linear momentum*** $\vec{p}$ "



$$\vec{p} = m\vec{v} \qquad (1)$$

m is *mass* and $\vec{v}$ is the *velocity* of considered object.

- This parameter allows to state the second law of Newton in a very ***useful form*** ;

$$\vec{F}_{Net} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt} \qquad (2)$$

2nd Law: "The net force acting on a particle is equal to the change rate of its linear momentum"

If the mass of object remains constant during the observation the expression (2) transforms to the classic form on Newton's law

$$\vec{F}_{Net} = m\frac{d\vec{v}}{dt} = m\vec{a} \qquad (3)$$

Note that relation (2) can describe the motion of an object even *if its mass changes* during the motion (for example in *reactive motion of rockets*) while the relation (3) cannot be used in those situations.

2] CONSERVATION OF LINEAR MOMENTUM DURING COLLISION

- In one of the first recorded experiments about the linear momentum, one found out that if two objects that constitute an *isolated system* (i.e. **net external force over them is zero**) stick together after collision (figure 1), the ***total linear momentum*** of the *system* of two objects *before* and *after* collision is the same, i.e.

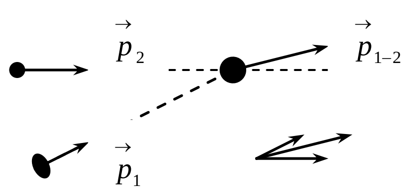


Figure 1

$$(\textit{before collision})\ \vec{p}_1 + \vec{p}_2 = \vec{p}_{1-2} \quad (\textit{after collision}) \qquad (4)$$

Further research confirmed that this is a particular application of a general mechanic's principle; *the conservation of linear momentum.*

- The ***collision*** is a type of interaction that happens during a ***real touch*** of particles (*like for billiard balls*) ***or*** *during* ***a no-touch repeal*** of particles (*like for electrical charges of the same sign*). Let's consider the collision of two particles with masses $\boldsymbol{m_1, m_2}$ moving at velocities $\vec{u}_1, \vec{u}_2$ ***before*** and $\vec{v}_1, \vec{v}_2$ ***after*** the collision (Fig.2). In general, after a collision, there is a change of linear momentum of each particle but their sum, i.e. the linear momentum of the system of the two particles *remains unchanged in specific situations*.

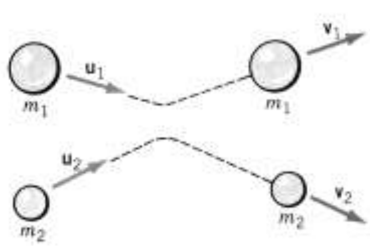


Figure 2

The Principle of Linear Momentum Conservation **states that:**
The ***total linear momentum*** of a ***system*** *of particles* remains ***constant*** if the ***net external force*** exerted on the system $\vec{F}_{Sys-Ext}$ **is zero.**

$$\Delta\vec{p} = \vec{p}_{fin} - \vec{p}_{in} = 0 \quad \text{or} \quad \vec{p}_{fin} = \vec{p}_{in} \qquad (5)$$

For the case of two particles' collision, relation (5) is written as:

$$m_1\vec{u}_1 + m_2\vec{u}_2 = m_1\vec{v}_1 + m_2\vec{v}_2 \qquad (6)$$

The vector relation (6) transforms into three scalar relations (7), when projected on axes of an Oxyz frame.

The three relations (7) show that, <u>*when*</u> the conservation of linear momentum <u>*holds on*</u>, each of its ***components in a Oxyz reference frame is independently conserved***.

$$\begin{aligned} m_1u_{1x} + m_2u_{2x} &= m_1v_{1x} + m_2v_{2x} \\ m_1u_{1y} + m_2u_{2y} &= m_1v_{1y} + m_2v_{2y} \\ m_1u_{1z} + m_2u_{2z} &= m_1v_{1z} + m_2v_{2z} \end{aligned} \qquad (7)$$

- One must not forget that, *in order to apply the principle of linear momentum conservation,* ***the <u>net external force</u> acting <u>on the system</u> must be <u>zero</u>.*** In the following, one derives formally the principle of linear momentum conservation for a system constituted by two particles with masses $\boldsymbol{m_1}$ and $\boldsymbol{m_2}$.

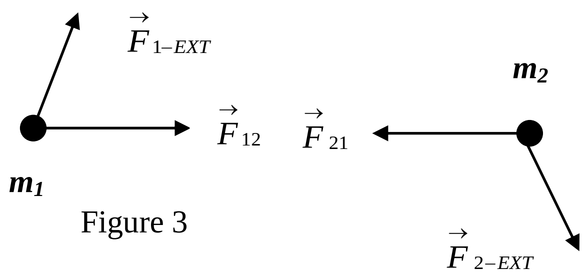


Figure 3

Two internal forces acting on this system are due to gravitational attraction: $\vec{F}_{12}$ exerted on particle "1" from the particle "2" and $\vec{F}_{21}$ exerted on particle "2" from particle "1". From the third law of Newton

$$\vec{F}_{21} = -\vec{F}_{12} \qquad (8)$$

By applying the equation (2) for the motion of particle (1) one get

$$\vec{F}_{1-NET} = \vec{F}_{1-EXT} + \vec{F}_{1-INT} = \vec{F}_{1-EXT} + \vec{F}_{12} = \frac{d\vec{p}_1}{dt} \qquad (9)$$

and similarly for the particle (2)

$$\vec{F}_{2-NET} = \vec{F}_{2-EXT} + \vec{F}_{2-INT} = \vec{F}_{2-EXT} + \vec{F}_{21} = \frac{d\vec{p}_2}{dt} \qquad (10)$$

By taking the side by side sum of relations (9) and (10) one get

$$\vec{F}_{Set-NET} = \vec{F}_{1-NET} + \vec{F}_{2-NET} = \vec{F}_{1-EXT} + \vec{F}_{12} + \vec{F}_{2-EXT} + \vec{F}_{21} = \vec{F}_{1-EXT} + \vec{F}_{2-EXT} = \vec{F}_{Set\_EXT} = \frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt}$$

(as $\vec{F}_{21} + \vec{F}_{12} = 0$ from relation 8) and finally

$$\vec{F}_{Set-EXT} = \frac{d(\vec{p}_1 + \vec{p}_2)}{dt} = \frac{d\vec{p}_{Set}}{dt} \qquad (11)$$

$\vec{F}_{Set-Ext} = \vec{F}_{1-Ext} + \vec{F}_{2-Ext}$ is the sum of external forces acting on two particles and $\vec{p}_{Set}$ is the <u>***sum***</u> of their <u>***linear momentums***</u>. Then, <u>*if the net external force acting over the set is zero*</u>, from relation (11) one gets

$$\frac{d\vec{p}_{Set}}{dt} = 0 \qquad \text{which means that} \qquad \vec{p}_1 + \vec{p}_2 = c^{te} \qquad (12)$$

In the case of many particles, $\vec{p}_{sys} = \sum_i \vec{p}_i$ and condition (12) becomes $\vec{p}_{sys} = \sum_i \vec{p}_i = c^{te}$ (13)

Those relations (12, 13) hold on as long as there is zero net external action over the system of particles.

<u>**Note:**</u> If $\vec{F}_{Set-Ext} \neq 0$ but it is a <u>***constant force***</u>, this force is a vector along a fixed ***direction*** in space and it has ***zero components*** over a ***plane perpendicular to its direction***. In this case, one may select the **Ox, Oy** axes on that plane and apply the two first relations of system (7). So, even though $\vec{F}_{Sys-Ext} \neq 0$, provided that it is a <u>***constant force***</u> (*i.e. fixed direction in space*)**,** the ***components*** of linear momentum of the **system *are conserved*** along the directions perpendicular *to* $\boldsymbol{\vec{F}_{Set-Ext}}$.

3] TYPES OF COLLISIONS

-The principle of linear momentum conservation is a basic principle that applies in all physics studies like mechanical collisions, explosions, reactive motion, light emission/absorption, nuclear radioactive decay and nuclear reactions,.... So, it is important to clarify the way one refers *to the term "**collisions**".*

- In physics, one uses the term collision when referring to a "*brief and strong interaction between two or more bodies*". What does one consider as a *brief time of interaction* ? The " *time interval of collision* Δt " depends on the considered phenomena. For common ***mechanical collisions*** (between balls, cars, people,..) the collisions last *from **0.001s** to **1 s***. A collision between ***elementary particles lasts*** for **~10^{-23}*s***. In the case of ***galaxies***, a collision lasts about ***several millions of years***.

- There are *two main types* of collisions; ***elastic*** and ***inelastic***. In *both cases*, if $\vec{F}^{system}_{net\_external} = 0$ ***during the interval of collision*** "Δt ", the ***linear momentum of the "system of colliding objects" is conserved*** $\overrightarrow{\Delta P}_{sys} = 0$.
In case of ***ELASTIC*** collisions, the ***TOTAL KINETIC energy of the system is CONSERVED,*** too.
So, for an ***elastic collision*** between ***two particles*** of an ***isolated system (i.e. $\vec{F}_{Net\_ext} = 0$***), one get two relations

$$m_1 \vec{u}_1 + m_2 \vec{u}_2 = m_1 \vec{\upsilon}_1 + m_2 \vec{\upsilon}_2 \quad \text{and} \quad \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = \frac{1}{2}m_1\upsilon_1^2 + \frac{1}{2}m_2\upsilon_2^2 \qquad (14)$$

Note: During an *elastic collision*, the *kinetic energy* of the system is transformed (*partially or completely*) into *potential elastic energy* but ***after the collision*** *(i.e. after Δt)* ***it is completely recovered into kinetic energy of the system.*** The real elastic collisions are common phenomena in atomic or nuclear physics. In everyday life there are no really elastic collisions. *Note that one uses often the elastic model as a first step approximation for collisions of steel or billiard balls, but they are not really pure elastic collisions.*

- In ***INELASTIC COLLISIONS*** the ***TOTAL KINETIC ENERGY*** of *the system* is ***NOT CONSERVED***.
During an inelastic collision, a part of kinetic energy of the system is lost. It get converted into *thermal* energy, *potential energy due to deformations*, *sound* energy and even *light* energy. It is important to underline that the converted part ***does not recover into kinetic energy of system after the collision****(i.e. after Δt).*
In a ***COMPLETELY INELASTIC COLLISION***, the bodies ***stick together*** after collision.

4] ELASTIC COLLISION OF TWO PARTICLES

-The figure 4 presents the collision of two particles with masses ***m_1, m_2*** and initial velocities $\vec{u}_1, \vec{u}_2$ along the same space direction that we have labeled as ***Ox*** axis. We will consider "*a central collision*" which leaves the motion of particles along the same space direction, i.e. their velocities after collision $\vec{v}_1, \vec{v}_2$ are along ***Ox*** axe, too.

So,
$$\vec{u}_1 = u_1 \vec{i};\_\vec{u}_2 = u_2 \vec{i};\_\vec{\upsilon}_1 = \upsilon_1 \vec{i};\_\vec{\upsilon}_2 = \upsilon_2 \vec{i} \qquad (15)$$

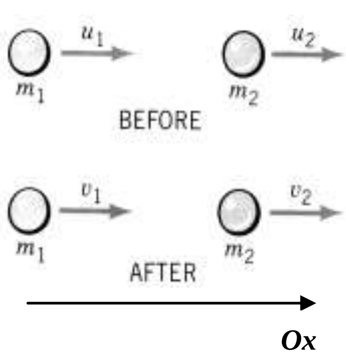


Figure 4

In figure, all velocities are drawn (for simplicity) along one sense. Note that there is collision if $u_1 > u_2 > 0$ or if $u_1 > 0$ and $u_2 < 0$ (which appears at algebraic values of velocities). As this is an **elastic collision**, both **conservation of linear momentum and kinetic energy apply**.
By projecting the relations at (14) on ***Ox*** axis, one gets:

$$m_1u_1 + m_2u_2 = m_1\upsilon_1 + m_2\upsilon_2 \rightarrow\rightarrow m_1(u_1 - \upsilon_1) = -m_2(u_2 - \upsilon_2) \qquad (16)$$

$$m_1u_1^2 + m_2u_2^2 = m_1\upsilon_1^2 + m_2\upsilon_2^2 \rightarrow\rightarrow m_1(u_1^2 - \upsilon_1^2) = -m_2(u_2^2 - \upsilon_2^2) \qquad (17)$$

Next, by writing eq. (17) in form $m_1(u_1 - v_1)(u_1 + v_1) = -m_2(u_2 - v_2)(u_2 + v_2)$ (18)
and dividing eq.(18) to eq.16 one gets $(u_1 + v_1) = (u_2 + v_2) \ggg (v_2 - v_1) = -(u_2 - u_1)$ (19)

The relation (19) shows that after a ***1-D*** ***central elastic collision,*** the ***magnitude*** of ***relative velocity*** of the ***second particle*** (often called " target ") ***versus the first particle remains the same but its direction is inverted.*** (see *the definition of relative velocity in 1-D at Galileo transformations*).

- ***Ex_1.*** *Find* υ_1, υ_2 *for the* ***central collision*** *of two particles with* ***equal mass*** *(i.e.* $m_1 = m_2 \equiv m$). In this case eq.(16) can be rewritten as $u_1 - v_1 = -(u_2 - v_2)$ or $v_1 + v_2 = u_1 + u_2$ (20)
By *adding/subtracting* relation (19) *to / from* (20) one can find that $v_2 = u_1\_and\_v_1 = u_2$ (21)

In particular, if before collision the *target* (***particle 2***) is at rest, i.e. $u_2 = 0;\_u_1 \neq 0$, it comes out that, after collision $\upsilon_1 = u_2 = 0$, i.e. the first particle stops moving and $\upsilon_2 = u_1$, i.e. the second particle starts moving at the velocity of the first particle before collision. One may see this situation often in billiard games.

Ex_2. *Find* υ_1, υ_2 *for the* ***central collision*** *of two particles with unequal masses (* $m_1 \neq m_2$) when the second particle (or the *target*) is ***at rest*** ($u_2 = 0;\_u_1 \neq 0$). In this case eq.(17) gives $m_1 u_1 = m_1\upsilon_1 + m_2\upsilon_2$ (22)
while the relation (19) transforms to $u_1 = \upsilon_2 - \upsilon_1$ (23)

The solution of system (22-23) gives $\upsilon_1 = \dfrac{(m_1 - m_2)u_1}{m_1 + m_2}$ and $\upsilon_2 = \dfrac{2m_1 u_1}{m_1 + m_2}$ (24)

The expressions (24) become simpler in the two following situations;

- when $m_1 \gg m_2$, *it comes out that* $\upsilon_1 \approx u_1\_and\_\upsilon_2 \approx 2u_1$ *i.e. the first particle follows moving at its initial velocity but the second one imparts at a speed the* ***double of*** u_1. This happens if a solid object with ***big mass*** "m_1" ***hits centrally*** a solid object with ***small mass*** "m_2".
This is the case of a golf ball motion after the club hits on it.
- *when* $m_1 \ll m_2$, *it comes out that* $\upsilon_1 \approx -u_1\_and\_\upsilon_2 \approx 0$ *i.e. the first particle reverses the sense but keeps the same magnitude of velocity.* This would correspond to the situation when a ping-pong ball hits centrally a bowling ball or a wall.

Ex_3. Non-central *elastic* collision of two particles with same mass "m" when the ***target is at rest***. This case corresponds to the situation shown in figure 5 (two billiard balls when one of them is initially at rest).

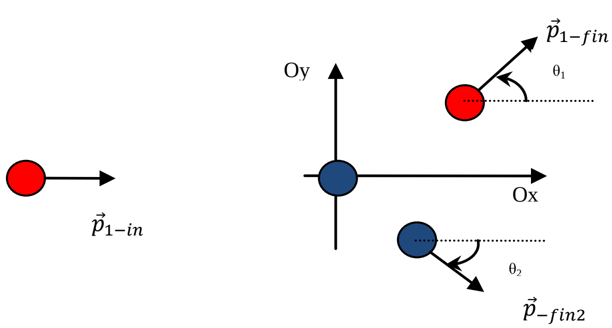


Figure 5 $(\theta_{1-2} = \theta_1 + \theta_2 = 90^0)$

In this case, linear momentum conservation gives

$$\vec{p}_{1-in} = \vec{p}_{1-fin} + \vec{p}_{2-fin} \quad (25)$$

By taking the square of both sides, one gets

$p_{1-in}^2 = p_{1-fin}^2 + p_{2-fin}^2 + 2\vec{p}_{1-fin} * \vec{p}_{2-fin}$ or

$$p_{1-in}^2 - (p_{1-fin}^2 + p_{2-fin}^2) = 2\vec{p}_{1-fin} * \vec{p}_{2-fin} \quad (26)$$

As an elastic collision, the kinetic energy is conserved

$$K_{1-in} = K_{1-fin} + K_{2-fin} \quad (27)$$

As $\frac{mv^2}{2} = \frac{m^2v^2}{2m} = \frac{p^2}{2m}$ one can rewrite (27) as

$\frac{p_{1-in}^2}{2m} = \frac{p_{1-fin}^2}{2m} + \frac{p_{2-fin}^2}{2m}$ or $p_{1-in}^2 = p_{1-fin}^2 + p_{2-fin}^2$

By substituting it to the left at (26) it comes out that

$$2\vec{p}_{1-fin} * \vec{p}_{2-fin} = 2p_{1-fin} * p_{2-fin} * \cos\theta_{1-2} = 0$$

Then, one get $\cos\theta_{1-2} = 0$, i.e. the two vectors of final linear momentum are perpendicular to each other (fig.5).

5] IMPULSE

- One names as " ***impulsive actions*** " the actions that last for a very ***short*** interval of ***time*** and studies them by use of a specific physical quantity, the ***vector of impulse***. In general, one says that there is an ***impulse applied*** on a particle if its ***linear momentum*** *changes (from* $\vec{p}_i$ *to* $\vec{p}_f$ *)* and defines the vector of ***impulse*** $\vec{I}$ as

$$\vec{I} = \vec{p}_f - \vec{p}_i = \Delta \vec{p} \qquad (28)$$

- Both, the *impulse* and *the net force* exerted on a particle can be related to the *changes* of *linear momentum.* So, there is a relation between them, too. From the " modern form " of the second law $\vec{F}_{Net} = \frac{d\vec{p}}{dt}$

one can get the differential expression

$$d\vec{p} = \vec{F}_{Net}\, dt \qquad (29)$$

The relation (28, 29) show that the ***impulse vector*** has a unit [***kg*m/s***] or [**N*s**] in SI system.

By taking the integral of relation (29) one can calculate the *finite change* of linear momentum $\Delta \vec{p}$ during an interval of time [t_i, t_f], i.e. the impulse applied on particle during this interval of time as

$$\vec{I} = \Delta \vec{p} = \int d\vec{p} = \int_{t_i}^{t_f} \vec{F}_{Net}\, dt \qquad (30)$$

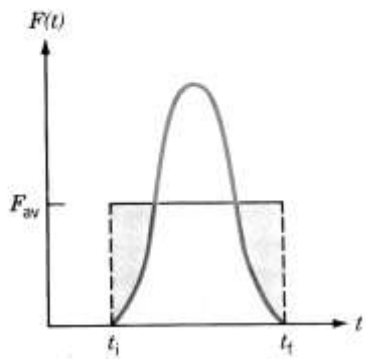


Figure 5

- In 1-D problems, $\vec{F}_{Net}, \vec{p}_{Fin}, \vec{p}_{In}$ have the same direction and (30) transforms

to the scalar expression

$$I = \Delta p = \int dp = \int_{t_i}^{t_f} F_{Net}\, dt \qquad (31)$$

Without going into details, we remind that the integral (31) is equal to the area under the graph ***F = F(t)***. Note that the relation (31) is valid for any type of force that changes in time, but one uses it *mainly for* ***impulsive forces***.

-An ***impulsive force acts during a short interval of time inside which it increases and decreases abruptly***. ***<u>During this short interval</u>, the action of impulsive force is much bigger*** *than all other forces exerted on the particle. S*o, one ***neglects*** *the effect of* ***other forces*** <u>*when an impulsive force is in action*</u>.

As $F_{Net} \approx F_{imp}$, then the integral (31) transforms to

$$I = \int_{t_1}^{t_2} F_{imp}\, dt \qquad (32)$$

This expression calculates the magnitude of impulse based *only on the area under the graph of* ***impulsive force*** ***F_{imp} (t)*** (typical shape is shown in figure 5). This step is known as the ***impulsive approximation***.

Example*; The impulsive force exerted on a ball* ***during the short interval*** *of* ***time*** *a player kicks a soccer* ball is **much *bigger than ball weight*** *and it is this force that decides about the change of linear momentum of the ball.* This ***impulsive force*** *gives the major contribution when calculating the integral (31,32).*

-The problem with impulsive forces is that, in general, *one does not know precisely the function* ***F_{imp} = F_{imp} (t).*** That's why one operates with an ***average constant*** *force* F_{Av} acting during the **same time interval** Δt (fig.5), such that the ***magnitude of its related impulse*** (*rectangle area*) is the same as the magnitude of impulse "***I*** " due to the impulsive force (*area under $F_{imp.}(t)$ graph*).

Then, one get the magnitude of impulse, i.e. the area calculated by (32) as

$$I = \Delta p = F_{Av}\Delta t \qquad (33)$$

- The presentation of ***impulse magnitude*** by the area under the graphs allows to figure out that the **same *change of linear momentum of a particle***(i.e. object) *can be achieved* (see fig. 6) :
a) by use of an impulsive force applied briefly or
b) by use of a "less impulsive" force applied for a longer interval of time.

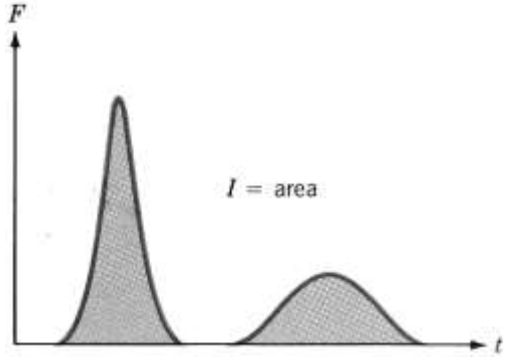


Figure 6

This kind of information is very important when:

- The application of a very large force for a short interval of time can create problems for the stability of the object (or destroy it).
- There is a limited flexibility about the production of large forces.
- A bit longer Δt interaction time does not affect essentially the expected effect.