LECTURE 2

1. THE SPACE RELATED PROPRIETIES OF PHYSICAL QUANTITIES

Physics uses physical parameters. In this course one will deal only[1] with scalar and vector parameters. Scalar parameters do not depend *on the space direction.* Vector parameters depend on *space directions*. *Ex: An insect moves on a plan(i.e. a 2D space);its* ***displacement*** *is a vector but the* ***travelled distance*** *is a scalar*.

2. VECTOR NOTATION AND OPERATIONS WITH VECTORS

One draws a vector as a ***directed line*** and labels it by a ***symbol*** (letters) covered by an ***arrow line.***
Ex.: $\vec{S}$_or_$\overrightarrow{AB}$_for_displacement_from_A_to_B ... $\vec{v}$_for_velocity, ... $\vec{F}$_for_force
The *length* of the vector *line* is *proportional* to the *vector* ***magnitude.*** The magnitude is noted by vector label or its absolute value sign (ex. υ or $|\vec{\upsilon}|$ for velocity). Note that the *magnitude* itself is a ***positive scalar***. The direction of line shows the ***direction*** of vector ***in space*** and the ***arrow*** shows its orientation ***sense***.

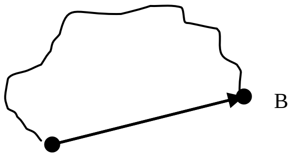


Figure 1

Equal displacement vectors. $\vec{AB} = \vec{CD}$
Two vectors are equal if they have; same **units**, equal **magnitudes** and same **orientations** in space.

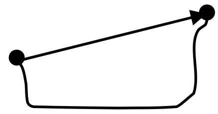


Figure 2

3. BASIC OPERATIONS WITH VECTORS

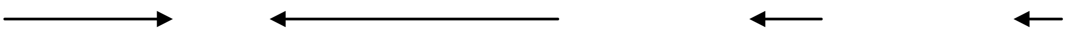
Multiplying a vector by a scalar*(with or without dimensions)*

-When ***multiplied/divided*** by a ***positive scalar*** *without dimensions*, ***only*** the vector'*s* ***length*** changes.

$\vec{A}[km]$ $2 * \vec{A}[km]$ $0.5 * \vec{A}[km]$ $\vec{A}/3[km]$

Figure 3 (same dimension)

-When ***multiplied/divided*** by a ***negative scalar*** *without dimensions*, the vector'*s* ***length*** changes **and** the orientation is inverted .



$\vec{A}[km]$ $-2\vec{A}[km]$ $-0.5\vec{A}[km]$ $-\vec{A}/3[km]$

Figure 4 (same dimensions)

-If the vector is ***multiplied/divided*** by a ***scalar with dimensions***, the same rules apply for orientation but the new vector has ***different units*** because it is another ***physical quantity*** (figure 5).

$\vec{\upsilon}[m/s]$ $\vec{AB}[m] = \vec{\upsilon}[m/s]*2[s]$ $\vec{F}_{el} = 10[N]$ $\vec{E}[N/C] = 10[N]/(-2[C])$

Displac.= veloc. * time   El. field = El. Force on negative charge /q

Figure 5 (different dimensions)

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[1] Some physical parameters called "tensors" depend in a more complicated way on direction in space.

Addition and substraction of two vectors

These two operations are **allowed only if** the vectors represent the **same physical quantities (*same dimensions)*** and have the ***same units.*** One **cannot add** a **velocity vector** to a **displacement vector.** One applies the **Tip –Tail** method for vector addition**; Shift one of vectors parallel to itself so that its tail fits to the tip of the other one.** The **vector sum has** the **tail located at the *free* tail** and **the tip located at the *free* tip.**

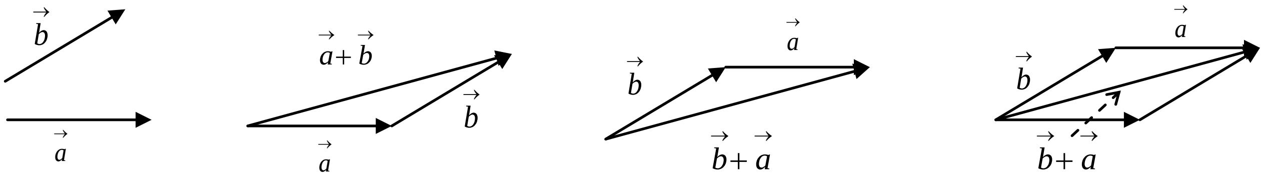


Figure 6.1 6.2 6.3 6.4

By comparing drawings in fig.6.2 and 6.3, 6.4 one can see that the vector $(\vec{a}+\vec{b})=(\vec{b}+\vec{a})$ (1)

-To subtract $\vec{b}$ **from** $\vec{a}$ **,** at first, one multiplies $\vec{b}$ by (-1) and gets the vector **-** $\vec{b}$ (fig7.2).

Next, one applies the rules for addition of vectors $\vec{a}$ and **-** $\vec{b}$ (Figure 7.1-4).

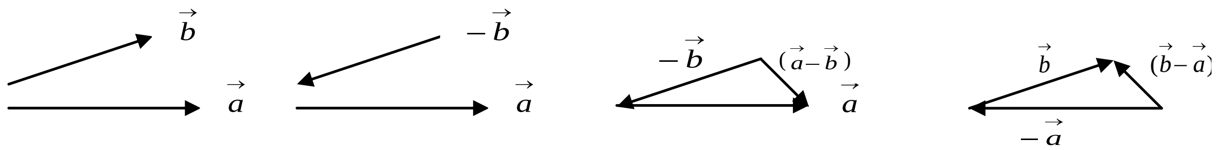


Figure 7.1 (7.2) (7.3) (7.4)

By comparing drawings in fig.7.3 and 7.4 one sees that the vector $(\vec{a}-\vec{b})=-(\vec{b}-\vec{a})$ (2)

Important note: In general, the magnitute of vector result ***is not equal*** to the sum or difference of magnitudes of added or substracted vectors. $\left|\vec{a}+\vec{b}\right| \neq \left|\vec{a}\right|+\left|\vec{b}\right|$ and $\left|\vec{a}-\vec{b}\right| \neq \left|\vec{a}\right|-\left|\vec{b}\right|$ (3)

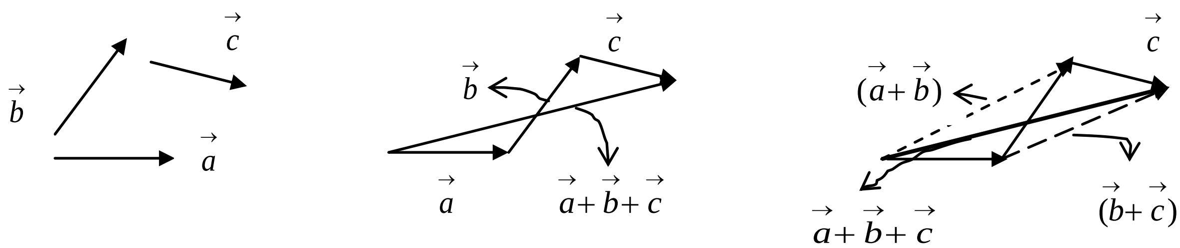


Figure 8.1 8.2 8.3

Note: The **addition of vectors is associative**: $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$ (4)

Fig. 8.2 and 8.3 show how to apply the tail-tip rule for three vectors.

Multiplying two vectors.

-The ***scalar (or dot *) product of two vectors*** is a ***scalar*** *quantity* ***with a new dimension***.

It is defined by the expression $\vec{a}*\vec{b} = |\vec{a}||\vec{b}|\cos\phi$ (5)

and can be positive, negative or zero depending on the angle between the vectors (see fig. 9.1-2).

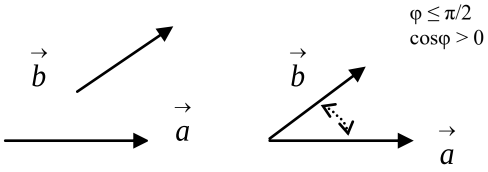
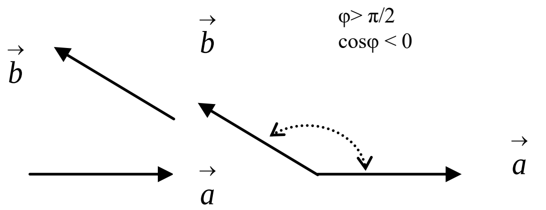


Figure 9.1



(9.2)

One can show that the ***scalar product*** is ***distributive***: $\vec{a}*(\vec{b}+\vec{c}) = \vec{a}*\vec{b}+\vec{a}*\vec{c}$ (6)

-The vector(cross) product of two vectors is a new vector, say $\vec{C}$**,** noted as $\vec{C} = \vec{A}\,x\,\vec{B}$ (7)

This vector has ***magnitude*** $|\vec{C}| = |\vec{A}||\vec{B}|\sin\phi$ (8) φ- is the smallest angle <u>from</u> $\vec{A}$ to $\vec{B}$.

One must refer to the angle ***0 ≤ φ ≤ π*** *because the* ***magnitude*** *of new* ***vector can not be negative.***

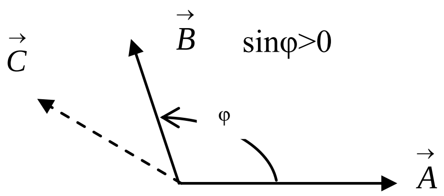


Figure 10

The direction of $\vec{C}$ is perpendicular to the *plane defined by vectors* $\vec{A}$ ***and*** $\vec{B}$***; its orientation is*** defined by convention, using the right-hand rule: *Curl the fingers of the right hand <u>from</u>* $\vec{A}$ *to* $\vec{B}$ *over the smallest angle and keep the thumb up. Thumb's direction gives the orientation of* $\vec{C}$**.** One may use also the rule of screw;

"The ***right-hand screw*** rule": If a right-hand screw turns from $\vec{A}$ ***to*** $\vec{B}$ *<u>along smallest angle between them</u>, the direction of its advancement is that of cross product vector* $\vec{C}$**.**

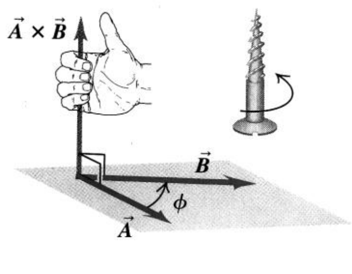
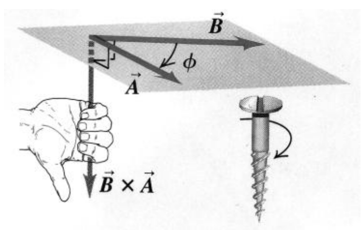


Figure 11.1



(11.2)

As seen from the figure 11.1-2, the ***vector product is not commutative***: $\vec{A}\,x\,\vec{B} = -\vec{B}\,x\,\vec{A}$ (9)

Unit vectors, coordinative systems and vector components.

- ***Unit vectors.*** A ***unit vector*** is a ***dimensionless*** vector with ***unit magnitude*** that serves only to *define a direction in space*. It is very useful for presentation of different physical vectors. In this course, we will use ***right-handed*** Cartesian systems (see Fig. 12) on which Ox, Oy, Oz directions are defined by three perpendicular unit vectors $\hat{i}, \hat{j}, \hat{k}$. By using the addition rule of vectors, one can present the vector $\vec{A}$

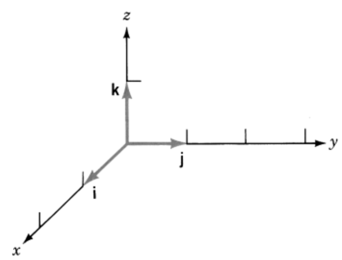


Figure 12

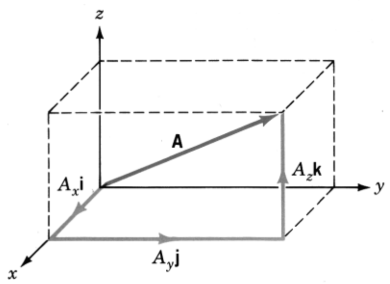


Figure 13

as $$\vec{A} = A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k} \quad (10)$$

$A_x\,\hat{i}$ is a vector along Ox axe with $|A_x|$ magnitude; A_x may be positive or negative, The <u>*dimension of the vector* $\vec{A}$ *goes with its components*</u> A_x, A_y, A_z . The same set of unit vectors $\hat{i}, \hat{j}, \hat{k}$ can be used to present different vector physical quantities.

Ex: $\vec{v}[m/s] = 5[m/s]\hat{\imath} + 2[m/s]\hat{\jmath} - 0.4[m/s]\hat{k}$; $\vec{F}[N] = 0.2[N]\hat{\imath} + 3.4[N]\hat{\jmath} + 1.4[N]\hat{k}$

- If $\vec{A}$ is a displacement, then A_x[m], A_y [m], A_z [m]; if $\vec{A}$ is velocity A_x[m/s], A_y [m/s], A_z [m/s] .

The three ***components*** A_x, A_y A_z in the reference frame Oxyz define fully the vector $\vec{A}$. By knowing the three components A_x, A_y , A_z , one may find the vector ***magnitude*** *via* Pythagora's theorem as

$$\left|\vec{A}\right| \equiv A = \sqrt{A_x^2 + A_y^2 + A_z^2} \quad (11)$$

- One may express the sum or the difference of two vectors through their components as follows:

If $\vec{C} = \vec{A} + \vec{B}$ then $C_x = A_x + B_x$; $C_y = A_y + B_y$; $C_z = A_z + B_z$ (12)

If $\vec{C} = \vec{A} - \vec{B}$ then $C_x = A_x - B_x$; $C_y = A_y - B_y$; $C_z = A_z - B_z$ (13)

If $\vec{C} = \vec{A}$ then $C_x = A_x$; $C_y = A_y$; $C_z = A_z$ (14)

The ***unit vector*** $\hat{A}$ along the direction of $\vec{A}$ has the **direction of** vector $\vec{A}$, the **magnitude 1** but it has **no dimension**. So, by using (10) and (11) one gets

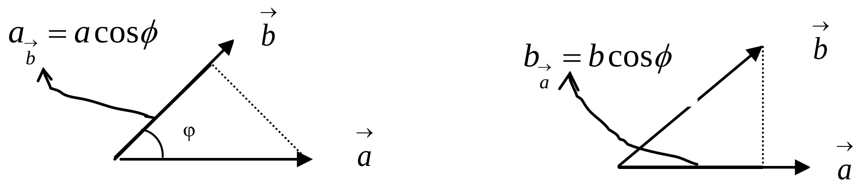
$$\hat{A} = \frac{\vec{A}}{\left|\vec{A}\right|} = \frac{\vec{A}}{\sqrt{A_x^2 + A_y^2 + A_z^2}} = \frac{A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k}}{\sqrt{A_x^2 + A_y^2 + A_z^2}} \quad (15)$$

Ex: If $\vec{a} = 1m\,\hat{i} + 2m\,\hat{j} + 3m\,\hat{k}$ then $\hat{a} = \dfrac{1m\,\hat{i} + 2m\,\hat{j} + 3m\,\hat{k}}{\sqrt{1^2 + 2^2 + 3^2}\,m} = \dfrac{1\hat{i} + 2\,\hat{j} + 3\,\hat{k}}{\sqrt{14}} = 0.267\,\hat{i} + 0.535\,\hat{j} + 0.802\,\hat{k}$

Expressing the dot and cross product by use of vector components in a Cartesiane system Oxyz

-***Dot product*** **(*)** $\vec{a} * \vec{b} = |\vec{a}||\vec{b}|cos\varphi = abcos\varphi = (a_{\vec{b}})b = a(b_{\vec{a}})$ (16)

Figure 14



One may figure out that if α, β, γ are the angles of a vector $\vec{A}$ to axes Ox,Oy,Oz , then one may calculate the scalar **components of vector** $\vec{A}$ along these axes as:

$$A_X = A\cos\alpha;\_A_y = A\cos\beta;\_A_Z = A\cos\gamma \quad (17)$$

Note that the dot product between each two unit vectors of a Cartesian frame Oxyz, gives or 1 or 0.

$$\hat{i}*\hat{i} = 1\cdot 1\cdot\cos 0 = 1; \qquad \hat{i}*\hat{j} = 1\cdot 1\cdot\cos\pi/2 = 0;\ \hat{i}*\hat{k} = 1\cdot 1\cdot\cos\pi/2 = 0$$

$$\hat{j}*\hat{j} = 1\cdot 1\cdot\cos 0 = 1; \qquad \hat{j}*\hat{i} = 1\cdot 1\cdot\cos\pi/2 = 0;\ \hat{j}*\hat{k} = 1\cdot 1\cdot\cos\pi/2 = 0 \quad (18)$$

$$\hat{k}*\hat{k} = 1\cdot 1\cdot\cos 0 = 1; \qquad \hat{k}*\hat{i} = 1\cdot 1\cdot\cos\pi/2 = 0;\ \hat{k}\cdot\hat{j} = 1\cdot 1\cdot\cos\pi/2 = 0$$

Consider two vectors $\vec{A}(A_x, A_y, A_z)$ and $\vec{B}(B_x, B_y, B_z)$ and their components in the same Oxyz frame.

$$\vec{A} = A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k} \quad \text{and} \quad \vec{B} = B_x\,\hat{i} + B_y\,\hat{j} + B_z\,\hat{k} \quad (19)$$

One may calculate the dot product $\vec{A}*\vec{B}$ by using the components (19) in the system of axes Oxyz.
By applying the results of dot product between unit vectors (18), one gets

$$\vec{A}*\vec{B} = (A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k})*(B_x\,\hat{i} + B_y\,\hat{j} + B_z\,\hat{k}) = A_xB_x + A_yB_y + A_zB_z \quad (20)$$

Important example: Calculate the ***magnitude*** of $\vec{A}$ by its components (A_x, A_y, A_z) in a frame Oxyz.

As $\vec{A}*\vec{A} = A\cdot A\cos 0 = A^2$ (*) and $\vec{A}*\vec{A} = A_xA_x + A_yA_y + A_zA_z = A_x^2 + A_y^2 + A_z^2$ (**)

by comparing (*) and (**) one finds out that $A^2 = A_x^2 + A_y^2 + A_z^2$ and $A = \sqrt{A^2} = \sqrt{A_x^2 + A_y^2 + A_z^2}$ (21)

-***Cross product*** **(x)** After introducing the ***concept of unit vector*** one can define the cross product by expression

$$\vec{A}\,x\,\vec{B} = (AB\sin\phi)\,\hat{n} \quad (22)$$

where $\hat{n}$ is a ***unit vector perpendicular to the plane*** defined by vectors $\vec{A}, \vec{B}$. Also, one may find the *components* of *vector* $\vec{A}\,x\,\vec{B}$ by their components; i.e. if $\vec{A}(A_x, A_y, A_z)$ and $\vec{B}(B_x, B_y, B_z)$

$$\vec{A}\,x\,\vec{B} = (A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k})x(B_x\,\hat{i} + B_y\,\hat{j} + B_z\,\hat{k}) = \left\{A_xB_x(\hat{i}\,x\,\hat{i}) + A_xB_y(\hat{i}\,x\,\hat{j}) + A_xB_z(\hat{i}\,x\,\hat{k})\right\} +$$
$$\left\{A_yB_x(\hat{j}\,x\,\hat{i}) + A_yB_y(\hat{j}\,x\,\hat{j}) + A_yB_z(\hat{j}\,x\,\hat{k})\right\} + \left\{A_zB_x(\hat{k}\,x\,\hat{i}) + A_zB_y(\hat{k}\,x\,\hat{j}) + A_zB_z(\hat{k}\,x\,\hat{k})\right\} \quad (23)$$

As *sin0^o=0 and sin π/2=1,* when applied between the unit vectors, the expressions (22) gives:

$$\hat{i}\,x\,\hat{i} = \hat{j}\,x\,\hat{j} = \hat{k}\,x\,\hat{k} = 1\sin 0 = 0 \quad \text{and} \quad \hat{i}\,x\,\hat{j} = (1\sin\pi/2)\hat{k} = \hat{k};\_\hat{i}\,x\,\hat{k} = -\hat{j};\_\hat{j}\,x\,\hat{k} = \hat{i}\ ..... \quad (24)$$

Taking into account the relations (24) one finds out that the expression (23) transforms to

$$\vec{A}\,x\,\vec{B}=\left\{A_xB_y(\hat{i}\,x\,\hat{j})+A_xB_z(\hat{i}\,x\,\hat{k})\right\}+\left\{A_yB_x(\hat{j}\,x\,\hat{i})+A_yB_z(\hat{j}\,x\,\hat{k})\right\}+\left\{A_zB_x(\hat{k}\,x\,\hat{i})+A_zB_y(\hat{k}\,x\,\hat{j})\right\}$$

$$\left\{A_xB_y\,\hat{k}-A_xB_z\,\hat{j}\right\}+\left\{-A_yB_x\,\hat{k}+A_yB_z\,\hat{i}\right\}+\left\{A_zB_x\,\hat{j}-A_zB_y\,\hat{i}\right\}= \qquad (25)$$

$$\left(A_yB_z-A_zB_y\right)\hat{i}+\left(A_zB_x-A_xB_z\right)\hat{j}+\left(A_xB_y-A_yB_x\right)\hat{k}$$

So, if $\vec{A}\,x\,\vec{B}\equiv\vec{C}(C_x,C_y,C_z)$ then

$$C_x=A_yB_z-A_zB_y;\_\_C_y=A_zB_x-A_xB_z;\_\_C_z=A_xB_y-A_yB_x \qquad (26)$$

Note: One may get quickly those components of cross product through the following determinant

$$\vec{A}\,x\,\vec{B}\equiv\vec{C}=C_x\,\hat{i}+C_y\,\hat{j}+C_z\,\hat{k}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix} \qquad (27)$$

<u>The application of vector algebra in translational&rotational equilibrium</u>

- One says that a ***force " $\vec{F}$ is applied on an object "*** if *this* ***object undergoes an action*** that ***tents***:

1] ***to move it*** along a given direction, if the object is *at rest*; or
2] ***to change its velocity vector*** (*magnitude, direction or both*), if the object is in movement.

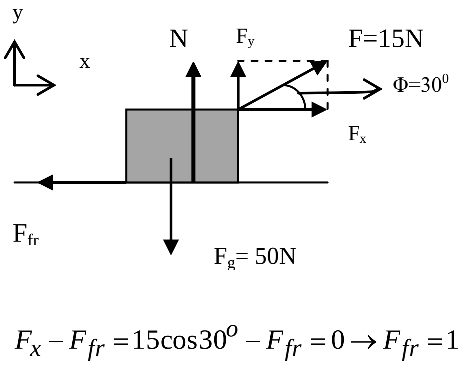
If the object ***<u>velocity does not change</u>*** even though one or several forces are acting on it, one says that it is in ***translational equilibrium***. *We will see that in this case the sum of all forces acting on the object is*

$$\sum_i \vec{F}_i=0 \qquad (28)$$

Example#1: A 5kg brick remains at rest(υ=*0*) on a table even though one applies a 15N force directed at 30^0 above the horizontal(fig.15). Find the *force of friction* and the *normal force* on the brick.

Note: The ***friction*** force is exerted *by the table* on the brick along its surface of contact. It has ***opposite direction*** to the direction of *possible* block shift. Fig. 15 shows the set of all forces acting on the brick.

Fig.15



There are four forces ***<u>exerted on the brick</u>***: $\vec{F}_g;\vec{N};\vec{F};\vec{F}_{fr}$.
As the ***brick is at rest*** one writes the relation (28) as

$$\sum_i \vec{F}_i=\vec{F}_g+\vec{N}+\vec{F}+\vec{F}_{fr}=0 \qquad (29)$$

Next, one selects axes ***Ox***, ***Oy*** as shown and project the vector eq.(29) on each of them. ($F_x=15N\cos 30^0\_and\_F_y=15N\sin 30^0$)

Ox: $F_x-F_{fr}=15\cos 30^o-F_{fr}=0\rightarrow F_{fr}=15\cos 30^o=13.0N$

Oy: $N-F_g+F_y=N-5*10+15\sin 30^o=0\rightarrow N=50-15\sin 30^o=42.5N$ (taking $g=10m/s^2$)

Example#2:Let's consider a bolt that *may rotate* around a ***fixed axe*** *passing through* its center (axe Oz in fig.16). Let's assume that *<u>the exerted action is not enough to start the rotation</u>*. So, it remains at rest. One says that this is an object *in the state of* ***rotational equilibrium***.

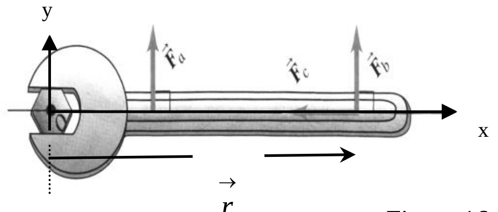


Figure 16

From our experience, we know that the ***rotational action*** (*we call* it ***torque*** action) on the bolt is larger if :
- the ***magnitude*** of exerted force F is bigger.
- the force is applied at bigger ***distance from rotation axe.***
- the force is applied at 90^o ***angle*** to wrench axe.

So, the ***rotation action is*** $\sim \left|\vec{F}\right|, \sim \left|\vec{r}\right|, \sim \sin(\vec{r},\vec{F})$. One defines the ***torque vector*** as $\vec{\tau} = \vec{r}\, x \vec{F}$ (30)

- The definition ***torque*** by the ***cross product*** does make sense: its magnitude is $\left|\vec{\tau}\right| = \left|\vec{r}\right|\left|\vec{F}\right|\sin(\vec{r},\vec{F})$

and it gives the orientation of rotational action, too. In bolt example (fig.16), the torque is directed along Oz $(\vec{r}\_along\_\hat{\imath}; \overrightarrow{F_b}\_along\_\hat{\jmath}\_and\_\vec{\tau}\_along\_\hat{k} = \hat{\imath}x\hat{\jmath})$ and rotational action is CCW.

Important note: *<u>The torque is **defined** always with respect to **a specific point**; an axe of possible rotation passes by this point</u>. When talking about the torque, one must precise the **torque with respect to "O- point"**. In rotation problems, one places the origin of coordinate system at O point.*

-As we will see, in ***rotational equilibrium***, ***the sum of all torques*** *acting on the object* ***is equal to zero***.

$$\sum_i \vec{\tau}_i = 0 \qquad (31)$$

By applying the condition (31) in the case of **bolt** at rotational equilibrium, we find that the action of the ***torque*** applied by ***wrench*** is cancelled by that of an "*internal*" ***torque*** due to friction of wall on the bolt.
These two torques have equal magnitude but opposite direction $\vec{\tau}_{ext} + \vec{\tau}_{\text{int}} = 0 \Rightarrow\Rightarrow \vec{\tau}_{\text{int}} = -\vec{\tau}_{ext}$ (32)

Exemple: *Find the* ***components of applied torque on wrench if*** $\left|\vec{r}\right| = 30cm = 0.3m$ *and the applied force 15N is directed along Oy (or ĵ). So, in Oxy system (see fig.16) the applied force has the components (0, 15N, 0) and the position r-vector has components (0.3m,0m,0m). By using the cross product general formula on gets*

$$\vec{\tau} = \tau_x\hat{\imath} + \tau_y\hat{\jmath} + \tau_z\hat{k} = \vec{r}x\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0.3m & 0 & 0 \\ 0 & 15N & 0 \end{vmatrix}$$
$$= \hat{\imath}(0*0 - 0*15N) - \hat{\jmath}\,(0.3m*0 - 0*0) + \hat{k}(0.3m*15N - 0*0)$$

So, $\tau_x = 0; \tau_y = 0; \tau_z = 4.5Nm$. The same result comes out by using the definition of magnitude by using the basic relation $\left|\vec{\tau}\right| = \left|\vec{r}\right|\left|\vec{F}\right|\sin(\vec{r},\vec{F}) = 0.3m \cdot 15N \cdot \sin(90^0) = 4.5Nm$.

One may get the same magnitude of torque even by applying a ***force*** with ***smaller*** magnitude but bigger $\left|\vec{r}\right|$ (***bigger lever arm).*** (*Ex*: Find ***F*** *magnitude* that applies the same torque if $\left|\vec{r}\right| = 0.4m$).