LECTURE 2

1. <u>THE SPACE RELATED PROPRIETIES OF PHYSICAL PARAMETERS</u>

Physics uses physical parameters. In this course one will deal only[1] with scalar and vector parameters. Scalar parameters do not depend *on the space direction.* Vector parameters depend on *space directions.* *Ex: An insect moves on a plan(i.e. a 2D space);its **displacement** is a vector but the **travelled distance** is a scalar.*

2. <u>VECTOR NOTATION AND OPERATIONS WITH VECTORS</u>

A vector as a ***directed line*** and one labels it by a ***symbol*** (letters) covered by an ***arrow line.***
Ex.: $\vec{S}\_or\_\overrightarrow{AB}\_for\_displacement\_from\_A\_to\_B \ldots \vec{v}\_for\_velocity, \ldots \vec{F}\_for\_force$
The *length* of the vector *line* is *proportional* to the *vector **magnitude.*** The magnitude is noted by vector

label or its absolute value sign (ex. v or $|\vec{v}|$ for velocity). Note that the *magnitude* itself is a ***positive scalar***. The direction of line shows the ***direction*** of vector ***in space*** and the ***arrow*** shows its orientation ***sense***.
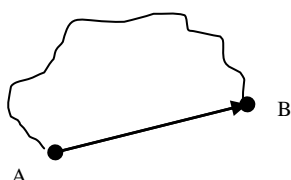

Equal displacement vectors. $\overrightarrow{AB} = \overrightarrow{CD}$
Two vectors are equal if they have;
same **units**, equal **magnitudes** and
same **orientations** in space.

Figure 1

Figure 2

3. <u>BASIC OPERATIONS WITH VECTORS</u>

Multiplying a vector by a scalar*(with or without dimensions)*

-When ***multiplied/divided*** by a ***positive scalar*** *without dimensions*, ***only*** the vector'*s* **length** changes.

$\vec{A}[km]$ $2*\vec{A}[km]$ $0.5*\vec{A}[km]$ $\vec{A}/3[km]$
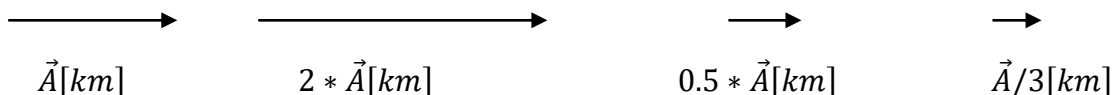
Figure 3 (same units)

-When ***multiplied/divided*** by a ***negative scalar*** *without dimensions*, the vector'*s* **length** changes **and** the orientation is inverted .

$\vec{A}[km]$ $-2\vec{A}[km]$ $-0.5\vec{A}[km]$ $-\vec{A}/3[km]$
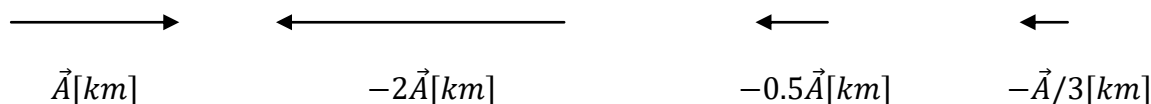
Figure 4 (same units)

-If the vector is ***multiplied/divided*** by a ***scalar with dimensions***, the same rules apply for orientation but the new vector has ***different units*** because it is another ***physical quantity*** (figure 5).
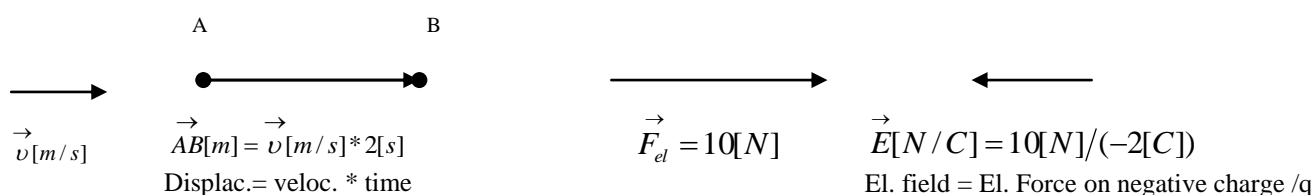

$\vec{v}[m/s]$ $\overrightarrow{AB}[m] = \vec{v}[m/s]*2[s]$ $\vec{F}_{el} = 10[N]$ $\vec{E}[N/C] = 10[N]/(-2[C])$
               Displac.= veloc. * time   El. field = El. Force on negative charge /q
Figure 5 (different units)

[1] Some physical parameters called "tensors" depend in a more complicated way on direction in space.

Addition and substraction of two vectors

These two operations are **allowed only if** the vectors represent the **same type of physical quantities** i.e. if they have the *same units.* One **cannot add** a **velocity vector** to a **displacement vector.**
One applies the **Tip –Tail** method for vector addition; **Shift one of vectors parallel to itself so that its tail fits to the tip of the other vector.** The **vector sum has** the **tail located at the** *free* **tail** and **the tip located at the** *free* **tip.**
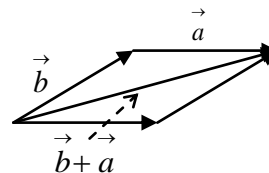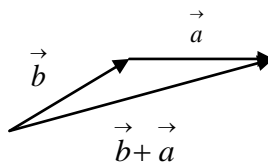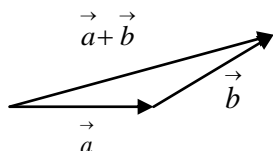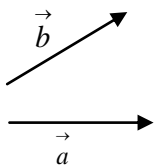

Figure 6.1   6.2   6.3   6.4

By comparing drawings in fig.6.2 and 6.3, 6.4 one can see that the vector $(\vec{a}+\vec{b})=(\vec{b}+\vec{a})$ (1)

-**To subtract** $\vec{b}$ **from** $\vec{a}$ **,** at first, one multiplies $\vec{b}$ by (-1) and gets the vector $-\vec{b}$ (fig7.2).

Next, one applies the rules for addition of vectors $\vec{a}$ and $-\vec{b}$ (Figure 7.1-4).
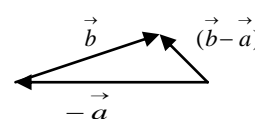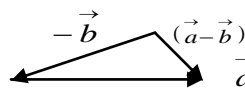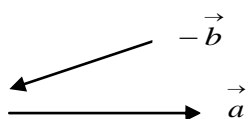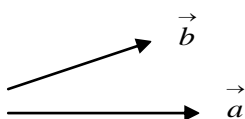

Figure 7.1   (7.2)   (7.3)   (7.4)

By comparing drawings in fig.7.3 and 7.4 one sees that the vector $(\vec{a}-\vec{b})=-(\vec{b}-\vec{a})$ (2)

Important note: In general, the magnitude of vector result *is not equal* to the sum or difference of magnitudes of added or substracted vectors. $\left|\vec{a}+\vec{b}\right|\neq\left|\vec{a}\right|+\left|\vec{b}\right|$ and $\left|\vec{a}-\vec{b}\right|\neq\left|\vec{a}\right|-\left|\vec{b}\right|$ (3)
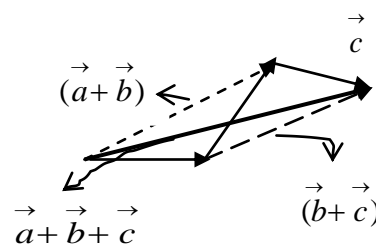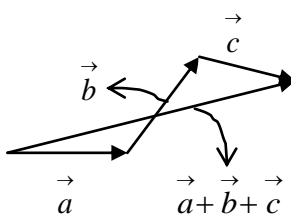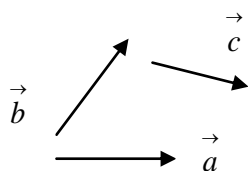

Figure 8.1   8.2   8.3

Note: The **addition of vectors is associative**: $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$ (4)

The figures 8.2 and 8.3 show how to apply the tail-tip rule for the sum of three vectors.

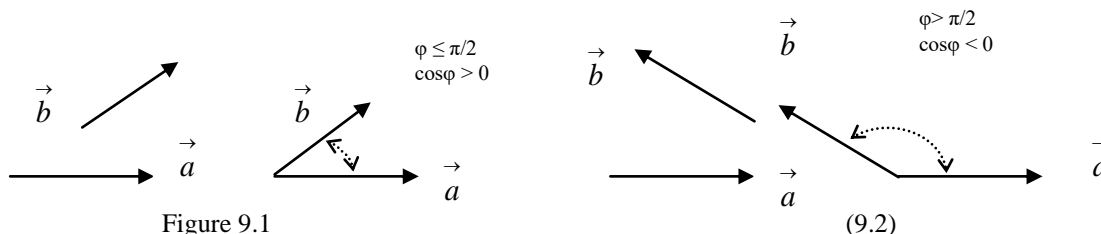## *Multiplying two vectors*.

-The **scalar (or dot *) product of two vectors** is a **scalar** *quantity* **with a new dimension**.

It is defined by the expression $\qquad\qquad\qquad\qquad\qquad \vec{a}*\vec{b}=\left|\vec{a}\right|\left|\vec{b}\right|\cos\phi$ (5)

and can be positive, negative or zero depending on the angle between the vectors (see fig. 9.1-2).

Figure 9.1 $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (9.2)

One can show that the **scalar product** is **distributive**: $\qquad \vec{a}*(\vec{b}+\vec{c})=\vec{a}*\vec{b}+\vec{a}*\vec{c}$ (6)

-**The vector(cross) product of two vectors is a new vector,** say $\vec{C}$**,** noted as $\quad \vec{C}=\vec{A}x\vec{B}$ (7)

This vector has **magnitude** $\left|\vec{C}\right|=\left|\vec{A}\right|\left|\vec{B}\right|\sin\phi$ (8) $\quad$ φ- is the smallest angle from $\vec{A}$ to $\vec{B}$.

One must refer to the angle $0\le\varphi\le\pi$ *because the **magnitude** of new **vector can not be negative.***
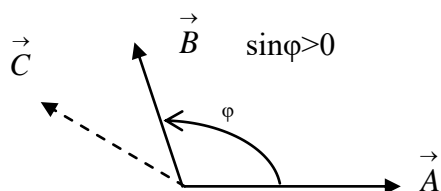

Figure 10

The direction of $\vec{C}$ is perpendicular to the *plane defined by vectors* $\vec{A}$ *and* $\vec{B}$ *; its orientation is* defined by convention, using the right-hand rule: *Curl the fingers of the* <u>right hand from</u> $\vec{A}$ *to* $\vec{B}$ *over the smallest angle and keep the thumb up. Thumb's direction gives the orientation of* $\vec{C}$**.** One may use also the rule of right hand screw;

"The **right-hand screw** rule": If a right-hand screw turns from $\vec{A}$ *to* $\vec{B}$ <u>along smallest angle between them</u>, the direction of its advancement is that of cross product vector $\vec{C}$**.**
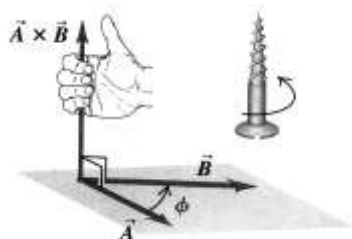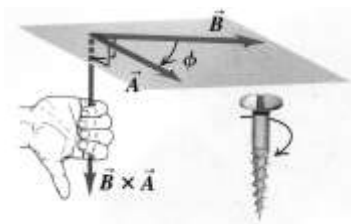

Figure 11.1 $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (11.2)

As seen from the figure 11.1-2, the **vector product is not commutative**: $\qquad \vec{A}x\vec{B}=-\vec{B}x\vec{A}$ (9)

3

Unit vectors, coordinative systems and vector components.

- *Unit vectors.* A *unit vector* is a *dimensionless* vector with *unit magnitude* that serves only to *define a direction in space*. It is very useful for presentation of different physical vectors. In this course, we will use *right-handed* Cartesian systems (see Fig. 12) on which Ox, Oy, Oz directions are defined by three perpendicular unit vectors $\hat{i}$, $\hat{j}$, $\hat{k}$. By using the addition rule of vectors, one can present the vector $\vec{A}$
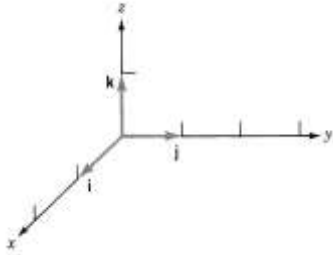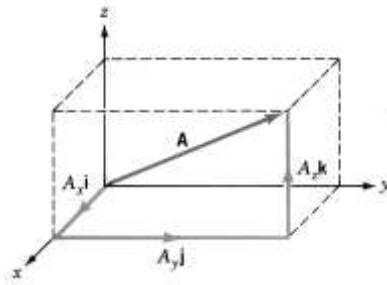
as $\qquad \vec{A} = A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k}$ (10)

$A_x\,\hat{i}$ is a vector along Ox axe with $|A_x|$ magnitude; A_x may be positive or negative,

The unit *of the vector $\vec{A}$ goes with its components* A_x, A_y, A_z. So, one uses the same set of unit vectors $\hat{i}$, $\hat{j}$, $\hat{k}$ to present different vector physical quantities.

Figure 12

Figure 13

Ex: $\vec{v}[m/s] = 5[m/s]\hat{i} + 2[m/s]\hat{j} - 0.4[m/s]\hat{k}$; $\vec{F}[N] = 0.2[N]\hat{i} + 3.4[N]\hat{j} + 1.4[N]\hat{k}$

- If $\vec{A}$ is a displacement, then A_x [m], A_y [m], A_z [m]; if $\vec{A}$ is velocity A_x[m/s], A_y [m/s], A_z [m/s] .

The three *components* A_x, A_y A_z in the reference frame Oxyz define fully the vector $\vec{A}$. By knowing the three components A_x, A_y, A_z, one may find the vector *magnitude* *via* Pythagora's theorem as

$$\left|\vec{A}\right| \equiv A = \sqrt{A_x^2 + A_y^2 + A_z^2} \tag{11}$$

- One may express the sum or the difference of two vectors through their components as follows:

If $\vec{C} = \vec{A} + \vec{B}$ then $C_x = A_x + B_x$; $C_y = A_y + B_y$; $C_z = A_z + B_z$ (12)

If $\vec{C} = \vec{A} - \vec{B}$ then $C_x = A_x - B_x$; $C_y = A_y - B_y$; $C_z = A_z - B_z$ (13)

If $\vec{C} = \vec{A}$ then $C_x = A_x$; $C_y = A_y$; $C_z = A_z$ (14)

The *unit vector* $\hat{A}$ along the direction of $\vec{A}$ has the **same** space **direction** as vector $\vec{A}$, the **magnitude 1** but it has **no dimension**. So, by using (10) and (11) one gets

$$\hat{A} = \frac{\vec{A}}{\left|\vec{A}\right|} = \frac{\vec{A}}{\sqrt{A_x^2 + A_y^2 + A_z^2}} = \frac{A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k}}{\sqrt{A_x^2 + A_y^2 + A_z^2}} \tag{15}$$

Ex: If $\vec{a} = 1m\,\hat{i} + 2m\,\hat{j} + 3m\,\hat{k}$ then $\hat{a} = \frac{1m\,\hat{i} + 2m\,\hat{j} + 3m\,\hat{k}}{\sqrt{1^2 + 2^2 + 3^2}\,m} = \frac{1\hat{i} + 2\hat{j} + 3\hat{k}}{\sqrt{14}} = 0.267\,\hat{i} + 0.535\,\hat{j} + 0.802\,\hat{k}$

4

Expressing the dot and cross product by use of vector components in a Cartesiane system Oxyz

-*Dot product* (*) $\vec{a} * \vec{b} = |\vec{a}||\vec{b}|cos\varphi = abcos\varphi = (a_{\vec{b}})b = a(b_{\vec{a}})$ (16)

Figure 14
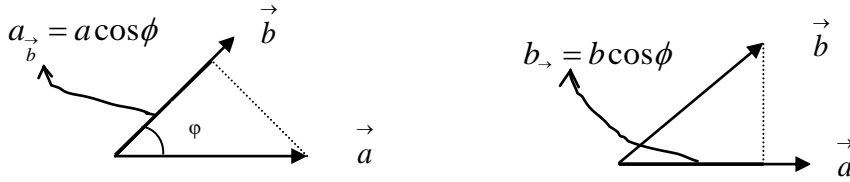
One may figure out that if α, β, γ are the angles of a vector $\vec{A}$ to axes Ox,Oy,Oz , then one may calculate the scalar **components of vector** $\vec{A}$ along these axes as:

$$A_x = A\cos\alpha; \_ A_y = A\cos\beta; \_ A_z = A\cos\gamma$$ (17)

Note that the dot product between each two unit vectors of a Cartesian frame Oxyz, gives or 1 or 0.

$$\hat{i} * \hat{i} = 1\cdot 1\cdot \cos 0 = 1; \qquad \hat{i} * \hat{j} = 1\cdot 1\cdot \cos\pi/2 = 0; \hat{i} * \hat{k} = 1\cdot 1\cdot \cos\pi/2 = 0$$

$$\hat{j} * \hat{j} = 1\cdot 1\cdot \cos 0 = 1; \qquad \hat{j} * \hat{i} = 1\cdot 1\cdot \cos\pi/2 = 0; \hat{j} * \hat{k} = 1\cdot 1\cdot \cos\pi/2 = 0 \qquad (18)$$

$$\hat{k} * \hat{k} = 1\cdot 1\cdot \cos 0 = 1; \qquad \hat{k} * \hat{i} = 1\cdot 1\cdot \cos\pi/2 = 0; \hat{k} \cdot \hat{j} = 1\cdot 1\cdot \cos\pi/2 = 0$$

Consider two vectors $\vec{A}(A_x, A_y, A_z)$ and $\vec{B}(B_x, B_y, B_z)$ and their components in the same Oxyz frame.

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \qquad and \qquad \vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$$ (19)

One may calculate the dot product $\vec{A} * \vec{B}$ by using the components (19) in the system of axes Oxyz. By applying the results of dot product between unit vectors (18), one gets

$$\vec{A} * \vec{B} = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) * (B_x \hat{i} + B_y \hat{j} + B_z \hat{k}) = A_x B_x + A_y B_y + A_z B_z$$ (20)

Important example: Calculate the ***magnitude*** of $\vec{A}$ by its components (A_x, A_y, A_z) in a frame Oxyz.

As $\vec{A} * \vec{A} = A \cdot A\cos 0 = A^2$ (*)   and $\vec{A} * \vec{A} = A_x A_x + A_y A_y + A_z A_z = A_x^2 + A_y^2 + A_z^2$ (**)

by comparing (*) and (**) one finds out that $A^2 = A_x^2 + A_y^2 + A_z^2$ and $A = \sqrt{A^2} = \sqrt{A_x^2 + A_y^2 + A_z^2}$ (21)

-*Cross product* "x" After introducing the ***concept of unit vector*** one can define the cross product by expression $\vec{A} x \vec{B} = (AB\sin\phi)\hat{n}$ (22)

where $\hat{n}$ is a ***unit vector perpendicular to the plane*** defined by vectors $\vec{A}, \vec{B}$. Also, one may find the *components* of *vector* $\vec{A} x \vec{B}$ by their components; i.e. if $\vec{A}(A_x, A_y, A_z)$ and $\vec{B}(B_x, B_y, B_z)$

$$\vec{A} x \vec{B} = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) x (B_x \hat{i} + B_y \hat{j} + B_z \hat{k}) = \left\{ A_x B_x (\hat{i} x \hat{i}) + A_x B_y (\hat{i} x \hat{j}) + A_x B_z (\hat{i} x \hat{k}) \right\} +$$

$$\left\{ A_y B_x (\hat{j} x \hat{i}) + A_y B_y (\hat{j} x \hat{j}) + A_y B_z (\hat{j} x \hat{k}) \right\} + \left\{ A_z B_x (\hat{k} x \hat{i}) + A_z B_y (\hat{k} x \hat{j}) + A_z B_z (\hat{k} x \hat{k}) \right\}$$

(23)

As *sin0°=0 and sin π/2=1,* when applied between the unit vectors, the expressions (22) gives:

$$\hat{i} x \hat{i} = \hat{j} x \hat{j} = \hat{k} x \hat{k} = 1\sin 0 = 0 \quad and \quad \hat{i} x \hat{j} = (1\sin\pi/2)\hat{k} = \hat{k}; \_ \hat{i} x \hat{k} = -\hat{j}; \_ \hat{j} x \hat{k} = \hat{i} \ .....$$ (24)

Taking into account the relations (24) one finds out that the expression (23) transforms to

$$\vec{A}\,x\,\vec{B} = \left\{ A_x B_y (\hat{i}\,x\,\hat{j}) + A_x B_z (\hat{i}\,x\,\hat{k}) \right\} + \left\{ A_y B_x (\hat{j}\,x\,\hat{i}) + A_y B_z (\hat{j}\,x\,\hat{k}) \right\} + \left\{ A_z B_x (\hat{k}\,x\,\hat{i}) + A_z B_y (\hat{k}\,x\,\hat{j}) \right\}$$

$$\left\{ A_x B_y \,\hat{k} - A_x B_z \,\hat{j} \right\} + \left\{ -A_y B_x \,\hat{k} + A_y B_z \,\hat{i} \right\} + \left\{ A_z B_x \,\hat{j} - A_z B_y \,\hat{i} \right\} = \qquad (25)$$

$$\left(A_y B_z - A_z B_y \right)\hat{i} + \left(A_z B_x - A_x B_z \right)\hat{j} + \left(A_x B_y - A_y B_x \right)\hat{k}$$

So, if $\vec{A}\,x\,\vec{B} \equiv \vec{C}(C_x, C_y, C_z)$ then

$$C_x = A_y B_z - A_z B_y;\ \_\_C_y = A_z B_x - A_x B_z;\ \_\_C_z = A_x B_y - A_y B_x \qquad (26)$$

Note: One may get quickly the components of cross product through the following determinant

$$\vec{A}\,x\,\vec{B} \equiv \vec{C} = C_x\,\hat{i} + C_y\,\hat{j} + C_z\,\hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \qquad (27)$$

The application of vector algebra in translational&rotational equilibrium

- One says that a *force "$\vec{F}$ is applied on an object "* if *this object undergoes an action* that *tents*:

    1] *to move it* along a given space direction, if the object is *at rest*;   or
    2] *to change its velocity vector* (*magnitude, direction or both*), if the object is in movement.

If the object *velocity does not change* even though one or several forces are acting on it, one says that it is in *translational equilibrium*. *We will see that in this case the sum of all forces acting on the object is*

$$\sum_i \vec{F}_i = 0 \qquad (28)$$

Example#1: A 5kg brick remains at rest($v=0$) on a table even though one applies a 15N force directed at 30^0 above the horizontal(fig.15). Find the *magnitudes* of *friction* and *normal* forces on the brick.

Note: The *friction* force is exerted *by the table* on the brick *along its surface of contact*. It has *opposite direction* to the *direction* of *possible* block *shift*. Fig. 15 shows the set of all forces acting on the brick.
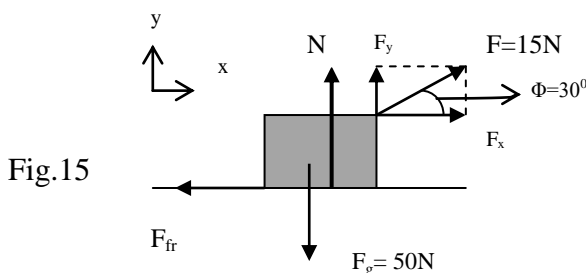

Fig.15

There are four forces *exerted on the brick*: $\vec{F}_g; \vec{N}; \vec{F}; \vec{F}_{fr}$.
As the *brick is at rest* one writes the relation (28) as

$$\sum_i \vec{F}_i = \vec{F}_g + \vec{N} + \vec{F} + \vec{F}_{fr} = 0 \qquad (29)$$

Next, one selects axes *Ox*, *Oy* as shown and project the vector eq.(29) on each of them. ($F_x = 15N\cos 30^0 \_and\_ F_y = 15N\sin 30^0$)

Ox: $F_x - F_{fr} = 15\cos 30^o - F_{fr} = 0 \rightarrow F_{fr} = 15\cos 30^o = 13.0N$

Oy: $N - F_g + F_y = N - 5*10 + 15\sin 30^o = 0 \rightarrow N = 50 - 15\sin 30^o = 42.5N$ (taking $g = 10m/s^2$)

Example#2:Let's consider a bolt that *may rotate* around a **fixed axe** *passing through* its center (axe Oz in fig.16). Let's assume that <u>the exerted action is not enough to start the rotation</u>. So, it remains at rest. One says that this is an object *in the state of* **rotational equilibrium**.
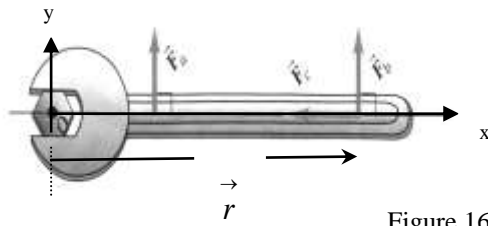

From our experience, we know that the **rotational action** (*we call* it **torque** action) on the bolt is larger if :
- the **magnitude** of exerted force F is larger.
- the force is applied at larger **distance from rotation axe.**
- the force is applied at 90° **angle** to wrench axe.

Figure 16

So, the **rotation action is** $\sim \left|\vec{F}\right|$, $\sim \left|\vec{r}\right|$, $\sim \sin(\vec{r}, \vec{F})$. One defines the **torque vector** as $\vec{\tau} = \vec{r} \, x \, \vec{F}$ (30)

- The **torque** definition by the **cross product** does make sense: its magnitude is $\left|\vec{\tau}\right| = \left|\vec{r}\right|\left|\vec{F}\right|\sin(\vec{r}, \vec{F})$

and it gives the orientation of rotational action, too. In bolt example (fig.16), the torque is directed along Oz $(\vec{r}\_along\_\hat{\imath}; \vec{F_b}\_along\_\hat{\jmath}\_and\_\vec{\tau}\_along\_\hat{k} = \hat{\imath} x \hat{\jmath})$ and rotational action is directed CCW.

Important note: <u>*The torque is **defined** always with respect to **a specific point "O"**; an axe of possible rotation passes by this point.*</u> *When talking about the torque, one must precise the **torque with respect to "O- point"**. In rotation problems, one places the origin of coordinate system at O point.*

-As we will see, in **rotational equilibrium**, **the sum of all torques** acting on the object **is equal to zero**.

$$\sum_i \vec{\tau}_i = 0 \qquad (31)$$

By applying the condition (31) in the case of **bolt** at rotational equilibrium, we find that the action of the **torque** applied by **wrench** is cancelled by that of an "*internal*" **torque** due to friction of wall on the bolt. These two torques have equal magnitude but opposite direction $\vec{\tau}_{ext} + \vec{\tau}_{int} = 0 \Rightarrow \Rightarrow \vec{\tau}_{int} = -\vec{\tau}_{ext}$ (32)

Exemple: *Find the **components** of applied torque on wrench if* $\left|\vec{r}\right| = 30cm = 0.3m$ *and the applied force*

15N is directed along Oy (or ĵ). So, in Oxy system (see fig.16) the applied force has the components (0, 15N, 0) and the position r-vector has components (0.3m,0m,0m). By using the cross product general formula on gets

$$\vec{\tau} = \tau_x \hat{\imath} + \tau_y \hat{\jmath} + \tau_z \hat{k} = \vec{r} x \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0.3m & 0 & 0 \\ 0 & 15N & 0 \end{vmatrix}$$

$$= \hat{\imath}(0*0 - 0*15N) - \hat{\jmath}(0.3m*0 - 0*0) + \hat{k}(0.3m*15N - 0*0)$$

So, $\tau_x = 0; \tau_y = 0; \tau_z = 4.5Nm$. The same result comes out by using the definition of magnitude by using

the basic relation $\left|\vec{\tau}\right| = \left|\vec{r}\right|\left|\vec{F}\right|\sin(\vec{r}, \vec{F}) = 0.3m \cdot 15N \cdot \sin(90^0) = 4.5Nm$.

One may get the same magnitude of torque even by applying a **force** with **smaller** magnitude but bigger $\left|\vec{r}\right|$ (**bigger lever arm).** (*Ex*: Find **F** **magnitude** that applies the same torque if $\left|\vec{r}\right| = 0.4m$).