MEASUREMENT AND UNCERTAINTY

Essentially, the measurement of any physical parameter is a process of comparison to a standard unit. The result of a measurement is a numerical value expressed in units of selected standard. This numerical value is always associated with an uncertainty.

Scale Reading Uncertainty. A student pours a liquid into a ***graduated cylinder with scale marks 1mL.*** To get an good reading, he moves his eye up/down (see Fig.1) to the level of meniscus in cylinder. If he reads somewhere between 255mL and 256mL, he would say that there is a volume of 255.5mL of liquid. Actually, he should report that " 255.5mL " is the ***best estimation value*** for the volume of liquid and there is an ***absolute uncertainty*** of +/-***0.5mL*** in this estimation***.*** This means that "*the true value*" of volume is "*for sure*" inside the ***uncertainty interval*** 255mL÷256mL and there is higher chances to be 255.5mL.

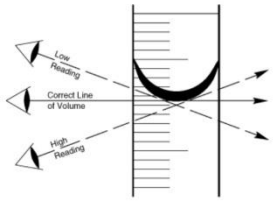


Fig.1 ***Parallax uncertainty***

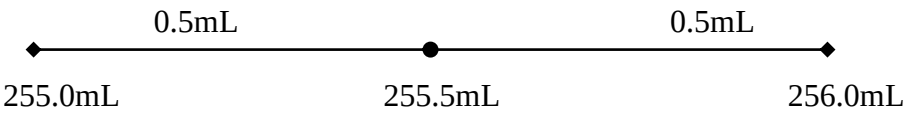


Fig.2 ***Interval of Uncertainty***

The right way of reporting the result for this measurement is V = (255.5 +/- 0.5) mL where " 255.5ml " is ***the best estimation*** and " 0.5mL " is the related ***absolute uncertainty***.
This type of absolute uncertainty is known as "***half minimum scale rule***".
The ***scale*** **reading uncertainty** is due to "**parallax uncertainty**" (or "*parallax error*") .
When reporting an absolute uncertainty as 0.5mL, one has assumed that the graduated cylinder is vertical and the student is reading the meniscus position while staying near by the graduated cylinder.

If the liquid is dangerous, the volume measurements must be done inside a "protective room" and the student cannot place his eye close to the graduated cylinder. In this situation, the parallax uncertainty becomes larger; if, he reads somewhere between 254mL and 258mL, he would report the result of volume measurement as V = (256.0 +/- 2.0) mL.
This example shows that the ***scale reading uncertainty*** **depends on the measurement conditions.**

In this type of measurement, it does make sense to accepts(***without measuring***) that the lower level of liquid fits to zero and the volume is the same as the scale reading. But, if one uses a "*mm scale*" ruler to measure the length of a muffler below car, one has to count the effect of reading uncertainty at both sides of muffler and the reading uncertainty would be much more than 0.5mm. *So, the* ***reading uncertainty is not*** *always equal the half* ***minimum scale*** *; it depends on the conditions of measurement, too.*

One defines the ***relative uncertainty*** *"ε"* as the ***portion****(in %)* ***of absolute uncertainty that corresponds to the unit of measured parameter.*** *So, for the measurement of liquid volume* $\boldsymbol{\varepsilon_V = \frac{\Delta V}{V_{Best}} * 100\%}$
When the student stands near by the cylinder the relative uncertainty is $\boldsymbol{\varepsilon_1 = 0.5/255.5*100\% = 0.2\%}$ *and when he stands outside* ***the*** "protective room the relative uncertainty is $\boldsymbol{\varepsilon_2 = 2.0/256.0*100\% = 0.8\%}$
Lower *is* ***ε - value, higher*** *is the* ***precision of a measurement.***
As $\varepsilon_1 < \varepsilon_2$, there is a higher measurement precision when the student stands close by graduated cylinder.

THREE CASES OF UNCERTAINTY

When dealing with uncertainty calculation of a given parameter one ***should start by distinguishing if it is directly measured*** *or it is* ***calculated*** *through some* ***measured parameters by use of an expression.*** *Rule: For a* ***direct measurement,*** *one* ***must repeat the same measurement several times.*** *As a result, there are two possible situations****:*** *or one gets* ***always*** *the* ***same reading*** *or one gets* ***different readings.***

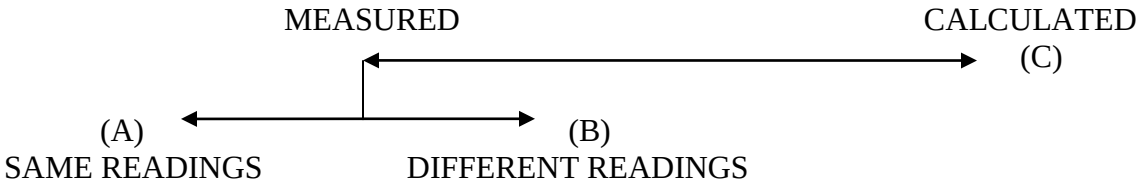


Fig.3 As a first step, one has to identify what case one is dealing with(A, B or C).

CASE "A" - You measure several times a parameter and you get the same numerical reading.

Example_1: You measure five times the width of a table by a meter stick which smallest unit is 1cm and read always 85cm. You may see that the reading is a little bit more or less than 85cm but you cannot read a more precise number because you cannot be precise about what portion of 1cm is the quantity "a little bit *more or less". So, you refer to "the half-scale rule" and take the* ***absolute uncertainty*** *equal to the half of the smallest unit available used for measurement (1cm); i.e. ΔL= ±0.5cm.*
*The result of measurement is reported as L= (85.0 ± 0.5)cm; ε=0.5/85.0*100=0.6% .*
The ***interval of uncertainty*** *for the measured parameter is (84.5; 85.5)cm.*

-If you use a meter stick with smallest unit available 1mm, you would get a more precise result but even in this case there is an uncertainty. Suppose that you will get always the reading L= 853mm. As there is always a "parallax reading uncertainty" (due to eye position) on both sides, one may get easily[1] *ΔL= ±1, ±2, and even ±3mm) depending on the measurement circumstances. The result of measurement would be reported as L= (853 ±1)mm or (853 ± 2)mm or (853 ± 3)mm. In three cases the* ***best estimation*** *for the table length is 853mm. If the* ***absolute uncertainty*** *of measurement is ΔL= ± 1mm, than the true length is inside the* ***uncertainty interval*** ***(****852, 854)mm and* $\varepsilon_1=1/853*100=0.12\%$;
if ΔL= ± 3mm, than the ***uncertainty interval*** ***(****850, 856)mm and* $\varepsilon_3=3/853*100=0.35\%$.

-Assume that, by using the same meter stick, you measure the length of a calculator and a room and find $\boldsymbol{L_{calc}= (14.0 \pm 0.5)cm}$ and $\boldsymbol{L_{room}= (525.0 \pm 0.5)cm.}$ Both measurements have the ***same absolute uncertainty*** $\boldsymbol{\Delta L= \pm 0.5cm}$ but the length of **room** is ***measured more precisely because*** the ***portion of uncertainty that belongs to the unit of measured length, i.e. the relative uncertainty is smaller.*** In this case

$\boldsymbol{\varepsilon\%} = \frac{\boldsymbol{\Delta L}}{\boldsymbol{B_{Best}}} * \mathbf{100}\%$ So, *one gets*

$\varepsilon_{calc} = \frac{0.5}{14} * 100\% = 3.57\% \approx 4\%$ ***and*** $\varepsilon_{room} = \frac{0.5}{525} * 100\% = 0.095\% \approx 0.1\%$.

This means that the room length is measured much ***more precisely (about 40 times).***

[1] Here one assumes no reading and no uncertainty for zero position. To get ΔL= ***±0.5mm one must apply special conditions (like a optic device for reading...)***

CASE "B" - You measure several times a parameter and you get different numerical readings.

Example_2: You drop an object from a window and expect (from theoretical calc.) it to hit ground after 2sec. To verify your expectation, you *measure* this *time* several times and record the following results;

1.99s, 2.01s, 1.89s, 2.05s 1.96s, 1.99s, 2.68s, 1.97s, 2.03s, 1.95s

Note: ***5 measurements*** is a ***minimum reasonable number of repeated data readings***, i.e. repeat five times the same measurement. Due to time restrictions in lab sessions, one may accept until ***three*** repeated measurements; *A result based on only one or two different measurements values is not reliable.*

As a first step one proceeds by: ***VERIFYING THE VALIDITY OF RECORDED DATA***

The drawing of graphs during lab measurements is a practical way to estimate quickly:

a) Whether the measurements confirm the expected behaviour predicted by the physical model.
b) If any of recorded data is measured in wrong way and must be excluded from further treatments.

To check out the data, one includes them in a graph (fig.4). From this graph one can see that:

a) The fall time seems to be *constant* and very likely ~2s. So, in general, these are acceptable data.
b) Only the seventh measure is too far from the others results and this *may be due to an abnormal* circumstance during its measurement. To eliminate any doubt, one ***excludes*** this value from the following data analysis. The remaining nine data are enough for following calculations.
1.99s, 2.01s, 1.89s, 2.05s, 1.96s, 1.99s, 1.97s, 2.03s, 1.95s.

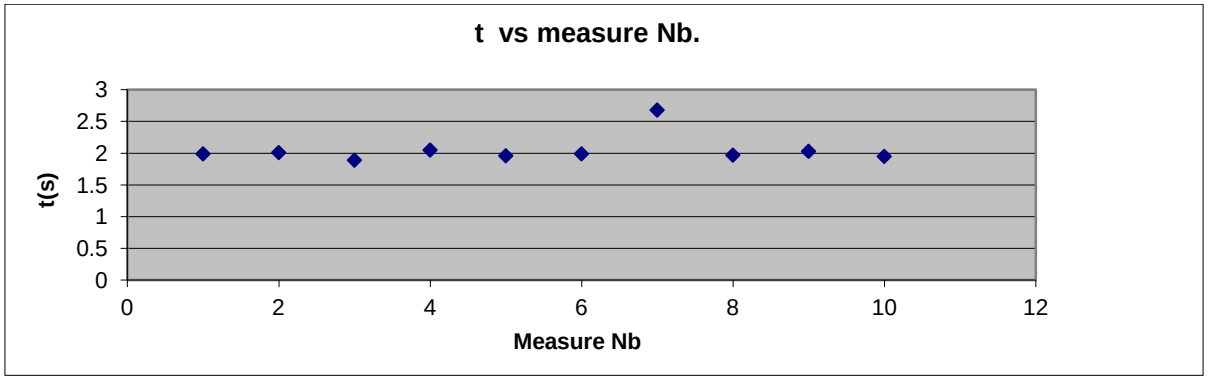


Fig.4

As a second step one proceeds by ***ORGANIZING RECORDED DATA IN A TABLE***

Include all data in a table organized in such a way that some cells be ready to include the results of uncertainty calculations. In this example, you are looking to estimate a single parameter "T", so you have to predict (at least) three cells; one for its *best estimation* and two for its *absolute&relative uncertainties*.

Table_1

t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{av}	Δt	$\varepsilon\%$
1.99s	2.01s	1.89s	2.05s	1.96s	1.99s	1.97s	2.03s	1.95s			

Then one follows with ***CALCULATIONS OF UNCERTAINTIES***

Keep in mind that the true value of parameter is unknown. So, one uses the recorded data to find an estimation of the true value and the uncertainty of this estimation.
In case ***"B"*** the ***best estimation*** of parameter is the ***average*** of recorded data and ***a good estimation*** of ***absolute uncertainty*** is the ***mean deviation*** of recorded data***.***

So, for the data collected in experiment_2

b.1) The *best estimation* for falling time is equal to the **average**.

$$t_{Av} = \frac{1}{9}\sum_{i=1}^{9} t_i = \frac{1}{9}[1.99 + 2.01 + 1.89 + 2.05 + 1.96 + 1.97 + 2.03 + 1.95] = 1.982 \cong 1.98s$$

b.2) One may get a good **estimation** for **absolute uncertainty** by using the ***spread of measured data from*** their ***average value.*** The ***mean deviation*** gives the average of this spread.

$$\Delta t = \frac{1}{n}\sum_{i=1}^{n}|t_i - t_{Av}| = \frac{1}{9}\sum_{i=1}^{9}|t_i - 1.982| = 0.0353 \cong 0.04s$$

This way, one would say that the **true value** *of fall time* is inside the ***uncertainty interval*** (1.94, *2.02*)***sec*** or between t_{min} =1. 94***s*** and t_{max} = 2.02***s*** with **best estimation** 1.98***s***. (after *rounding off*)

The result is reported as t = (1.98 +/- 0.04)sec

b.3) *Another (**statistically better**) estimation of spread* is the "**standard[2] deviation**" of data.

Based on our example data we get $\sigma(t) = \sqrt{\frac{\sum_{i=1}^{9}(t_i - t_{Av})^2}{n-1}} = \sqrt{\frac{\sum_{i=1}^{9}(t_i - 1.982)^2}{9-1}} = 0.047s$

After rounding off, the result is reported as ***t = (1.98 +/- 0.05)sec***

For *estimation of data spread*, a ***larger interval of uncertainty*** means a more "***conservative estimation***" but in the same time a ***more reliable estimation***. That's why the ***standard deviation*** is a better estimation for the ***absolute uncertainty***. Note that we get Δt = ***+/- 0.05s*** when using the ***standard deviation*** and Δt = ***+/- 0.04s*** when using the ***mean deviation.*** *Also, **the relative uncertainty** calculated from the **standard deviation*** is larger. In our example the ***relative uncertainty*** of measurements is

$\varepsilon = \frac{\sigma(t)}{t_{Av}} * 100\% = \frac{0.047}{1.982} * 100\% = 2.4\%$ when using the ***standard deviation***

and $\varepsilon = \frac{\sigma(t)}{t_{Av}} * 100\% = \frac{0.035}{1.982} * 100\% = 1.8\%$ when using the ***mean deviation***

The relative uncertainty 2.4% means lower precision of measurements but it is more reliable estimation.

Important: The absolute and relative uncertainty can never be zero.

[2] The standard deviation can be calculated direct in Excel and in many calculators.

-Assume that you **repeat** five times a given measurement and you read all times the **same value** of ***X***. So, by applying the rules of case "B" you may report $\mathbf{X_{best}} = X_{Av} = 5X/5 = X$ ***and*** $\Delta X_b = 0$. But this way you forget that, for each measurement, there is an uncertainty of type ***"A"*** and this means that there is at least $\Delta X_A = ½$ ***(smallest scale unit).*** This example shows that, in general, the absolute uncertainty of ***directly measured*** parameters should be calculated as $\Delta X = \Delta X_A + \Delta X_B$

Note_1 : In those cases where $\Delta X_B >> \Delta X_A$ ***one may simply take*** $\Delta X = \Delta X_B$
In exemple_2, the time is measured with 2 decimals. This means that $\Delta X_A = \Delta t_A = 0.01/2 = 0.005s$
As $\Delta X_B = \Delta t_B = 0.05s$ *is* ***ten times bigger one may*** *neglect* $\Delta t_A = 0.005s$ and get $\Delta t = \Delta t_B = 0.05s$.
But ***if*** $\Delta t_B = 0.01s$ *and* $\Delta t_A = 0.005s$ *one should not neglect* $\Delta t_A = 0.005s$ *because it is* ***50%*** *of* Δt_B.
So, one must calculate the absolute uncertainty as $\Delta t = 0.01 + 0.005 = 0.015s$

Note_2**:** When measuring a quantum parameter " ***X*** " one records "*a spectrum or a distribution of values*" for this parameter. This is a situation where $\Delta X_B >> \Delta X_A$ ***and one takes*** $\Delta X = \Delta X_B = \sigma(X)$

Note_3: In college labs one considers that a measurement has a *good precision* if the ***relative uncertainty*** is $\varepsilon < 10\%$. If the relative uncertainty is $\varepsilon > 10\%$, one may proceed by:

a) *Cancelling* any particular data "***shifted too much from the best estimation value***" ;
b) *Increasing* the number of recorded data by repeating more times the measurement;
c) *Improving* the measurement procedure.

CASE "C" **Estimation of Uncertainties for Calculated Parameters (Uncertainty propagation)**

Very often, one uses the experimental data recorded for several parameters and a mathematical expression to estimate the value of another parameter of interests (POI). As the measured parameters are estimated with an uncertainty, it is clear that the estimation of POI with have uncertainty, too.
The calculation of ***best estimation for POI*** is based on the ***best estimations of measured parameters*** and the ***formula*** that relates POI with measured parameters. The ***uncertainty of POI*** estimation is calculated by using the ***Max_Min method***. This method calculates the limits of uncertainty interval, POI_{min} & POI_{max} by using the formula relating POI with measured parameters and refers to the combination of their limiting values in such a way that the result be the smallest or the largest possible.

Example_3a. To measure the volume of an irregular shape solid, one starts by filling a graduated cylinder with water and records the volume reading, say V_1= 25.0mL. Next, one plunges the solid in water and records the new the volume reading, say V_2 = 125.5mL and calculates the solid volume as $V_{solid} = V_2 - V_1$
Those data mean that *minimum scale mark* is 1mL; so, V_1= (25.0+/-0.5)mL and V_2= (125.5+/-0.5)mL.
Then, one get $V_{best\text{-}solid}$ = 125.5 - 25.0 = 100.5mL; $V_{max\text{-}solid}$ = (125.5 + 0.5) - (25.0 - 0.5) = 101.5mL;
$V_{min\text{-}solid}$ = (125.5 - 0.5) - (25.0 + 0.5) = 99.5mL; $\Delta V = (V_{max\text{-}solid} - V_{min\text{-}solid})/2$= (101.5 - 99.5) /2 =1.0mL
Finally, one get V_{solid} = (100.5+/-1.0)mL and $\varepsilon\%$ = 1.0/100.5*100% = 0.99% ≈ 1%

Example_3b. To find the volume of a rectangular pool with constant depth, one uses a one meter stick to measure its length L, width W and depth D. Next, one calculates the volume by formula V = L * W * D. Assume that the measurement results are L = (25.5 ± 0.5)m, W = 12.0 ±0.5m, D = 3.5 ±0.5m. In this case the best estimation for the volume is $V_{best} = 25.5*12.0*3.5 = 1071.0\ m^3$. This estimation of volume is associated by an uncertainty calculated by Max-Min methods as follows
$V_{min} = L_{min}*W_{min}*D_{min} = 25*11.5*3 = 862.5m^3$ ***and*** $V_{max} = L_{max}*W_{max}*D_{max} = 26*12.5*4 = 1300.0m^3$

So, the ***uncertainty interval*** for the volume of pool is (862.5, 1300.0) and the ***absolute uncertainty*** *is* $\Delta V = (V_{max}-V_{min})/2 = (1300.0 - 862.5)/2 = 218.7 m^3$ while the **relative error** is $\varepsilon_V = \frac{218.7}{1071.0} * 100\% = 20.42\%$

When applying the sig. figure rules, one gets V_{best} =1100 m^3; ΔV= 220m^3; ε_V =20.%

Note_1: When applying the Max-Min method to calculate the uncertainty, one must pay attention to the mathematical expression that relates POI to the measured parameters.

Example_4 - You ***measure*** the ***period*** of an oscillation and you use it to ***calculate*** the ***frequency*** (POI). As $f = 1/T$, $f_{best} = 1/T_{av}$ *the* ***max-min method*** *gives* $f_{min} = 1/T_{max}$ *and* $f_{max} = 1/T_{min}$

Example_5 - *If* $z = x - y$, $z_{best} = x_{av} - y_{av}$ *and* $z_{MAX} = x_{MAX} - y_{MIN}$ *and* $z_{MIN} = x_{MIN} - y_{MAX}$.

Note_2. Use the *best estimations of parameters in the expression* to calculate the *best estimation for POI. If they are not available* one may use POI_{middle} as the best estimation for POI

$$POI_{middle} = \frac{POI_{MAX} + POI_{MIN}}{2}$$

Be aware though, that POI_{middle} *is not always equal to* ***best estimation*** of **POI.** So, the pool volume V_{middle}= (1300+862.5)/2 =**1081.3m^3** is different from V_{best} **=1071.0 m^3**(but both are rounded off as 1100).

***HOW TO PRESENT THE RESULT OF UNCERTAINTY CALCULATIONS*?**

You must provide the **best estimation,** the **absolute uncertainty** and the **relative uncertainty.** So, for the last example, the result of uncertainty calculations should be presented as follows:
V= (1100 ± 220) m^3 , ε% = (220/1100)*100% = 20%

Note: Absolute u*ncertainties* must be ***quoted*** to the ***same number of decimals*** ***as*** the **best estimation**. The use of *scientific notation* helps to prevent confusion about the number of significant figures.

Example_6: If calculations generate, say A = (0.03456789 ± 0.00245678)m
This should be presented after being rounded off (leave 1 or 2 digits after decimal point):
A = (3.5 ± 0.2) * 10^{-2}m or A = (3.46 ± 0.25) * 10^{-2}m

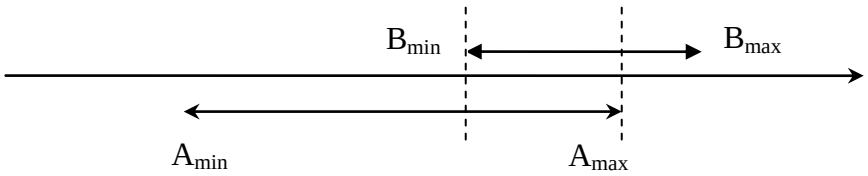
HOW TO CHECK WHETHER TWO QUANTITIES "A" and "B" ARE EQUAL?

This question appears essentially in two situations:
1.One measures the ***same parameter*** by two ***different methods*** and want to verify if the results are equal.
2.One uses measurements to ***verify*** if ***a theoretical expression*** is right.

In the first case, one have to compare the estimations $A \pm \Delta A$ and $B \pm \Delta B$ for the " two parameters ". The second case can be transformed easily to the first case by noting the measurement as ***A*** and the expression result as ***B***. Then, the procedure is the same.

Example: One wants to verify if the thins lens equation ***1/p + 1/q = 1/f***. So, one notes ***1/p + 1/q = A*** and ***1/f = B*** , records data for A and B, draws their uncertainty intervals and applies the following rule:
Rule: One considers that the quantities ***A*** and ***B*** are ***equal***[3] if their ***uncertainty intervals overlap***.

Fig.5



WORK WITH GRAPHS

One uses the graphs to check the theoretical expressions or to find the values of physical parameters.
Example_7; Theoretical study of simple pendulum oscillation shows that its period is $T = 2\pi\sqrt{L/g}$ and one wants to verify this experimentally. To simplify the procedure, one prefers to get a linear relationship between two parameters one can measure; in this case period T and length L. So, one squares both sides, get $T^2 = \frac{4\pi^2}{g} * L$ and by noting $\mathbf{T^2 = y}$, $\mathbf{L = x}$ gets the linear expression $\mathbf{y = a*x}$ where $\mathbf{a = 4\pi^2/g}$.
At first, one has to verify experimentally if there is such a relation between T^2 and L. Note that if this is verified, one can use the experimental value of "**a**" to calculate the free fall constant value "$\mathbf{g = 4\pi^2/a}$".

- Assume that after measuring the period for a given pendulum length several times, calculated the best values and uncertainties for ***y(=T²)*** and repeated this for a set of different values of length x(L=1,...,6m), one gets the data shown in table No 2. At first, one draws the graph of best data (X, Y_{best}). One can see that they are aligned on a straight line, *as expected*. Next, one uses Excel to find the best linear fitting line for these data and makes this line to pass from (x = 0, y = 0) because this is predicted from the theoretical formula. One gets a straight line with $\boldsymbol{a_{best} = 4.065}$. Using the theoretical formula one calculates the best estimation for $\boldsymbol{g_{best} = 4\pi^2/a_{av} = 4\pi^2/4.065 = 9.70}$ which is not far from the ***expected value 9.8***. Next, one adds the uncertainties in the graph(that appear as **error bars**) and draws the best linear fitting with maximum /minimum slope that pass by origin. From those graphs one gets $\boldsymbol{a_{min} = 3.635}$ and $\boldsymbol{a_{max} = 4.202}$. So, $\mathbf{g_{min}} = 4\pi^2/a_{max} = 4\pi^2/4.202 = \mathbf{9.38}$ and $\mathbf{g_{max}} = 4\pi^2/a_{min} = 4\pi^2/3.635 = \mathbf{10.85}$

Table_2

X	Y(best)	ΔY (+/-)	Ymin	Ymax	Max. Slope	Min. Slope
1	4	1.5	2.5	5.5	4.202	3.635
2	8.3	1.8	6.5	10.1		
3	11.8	1.3	10.5	13.1	P1 (1; 1.5)	P1 (1; 5.5)
4	17	1.6	15.4	18.6	P2 (6; 25.5)	P2 (6; 21.5)
5	21	1.1	19.9	22.1		
6	23.5	2	21.5	25.5		

This way, by using the graphs one has:
1- ***proved experimentally*** that the ***theoretical relation*** between ***T*** and ***L*** is right.
2- found out that the measurements are ***accurate*** because the ***uncertainty interval*** (9.38, 10.85) for "**g**" does *include the officially accepted value* $\boldsymbol{g = 9.8 m/s^2}$
3- found the ***absolute uncertainty*** $\Delta g = (10.85-9.38)/2 = 0.735 m/s^2$

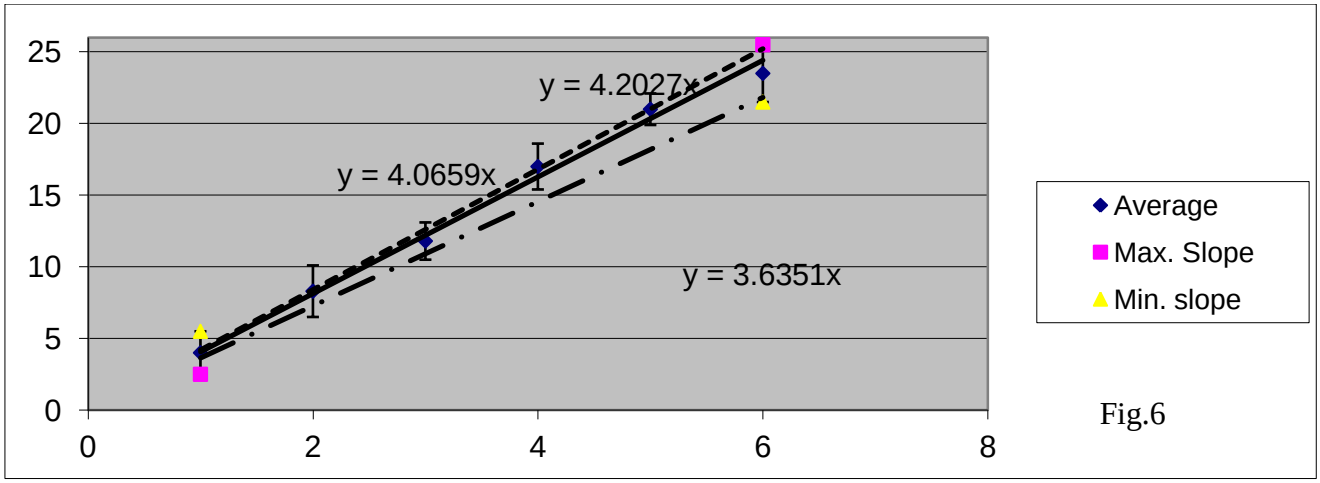
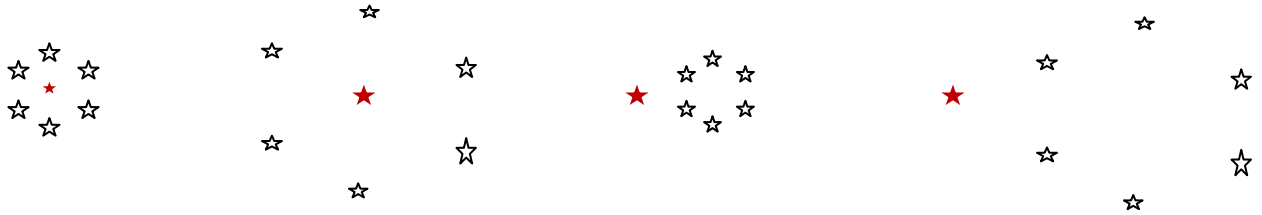


Fig.6

ABOUT THE ACCURACY AND PRECISION

- Understanding the difference between **accuracy** and **precision** by use of hits distribution in a Dart's play.



Accurate ***Good precision*** | ***Accurate*** ***Low precision*** | ***Inaccurate*** ***Good precision*** | ***Inaccurate*** ***Low precision***

- As a rule, before using a device (or method) for measurements***, one should make sure that the device*** (or method) ***produces accurate results in the range of expected values*** for the measured parameter**.** This is an obligatory step known as ***the calibration procedure.*** During a calibration procedure one records a set of data for a known parameter and makes sure that ***the result of measurements is accurate***. One considers that a calibrated device (or method) produces accurate results *inside calibration range* any time it is used.

In principle, the *result of an experiment is accurate* if the "***best estimation***" is equal to the " *officially accepted value*". In practice one considers that a measurement is "*accurate*" if the " *officially accepted value*" falls ***inside*** *the interval of uncertainty* of measured parameter; *otherwise the result is inaccurate.*

The quantity $\varepsilon_{accu} = \frac{|C_{Best} - C_{off}|}{C_{off}} x100$ ***%*** ***(often ambiguously named as error) gives the relative shift of best estimation from the officially accepted value*** C_{off}***. It is clear that the accuracy is higher when*** ε_{accu} ***is smaller. But, the measurement is inaccurate if*** $\varepsilon_{accu} > \varepsilon$ ***(relative uncertainty of measurement).***

Remember that relative uncertainty $\varepsilon = \frac{\Delta C}{C_{Best}} x100$ ***% is different from*** ε_{accu}***.***

Note: For a big number of measured data and accurate measurement, the best estimation should fit to the expected value of parameter and ε_{accu} ***should be practically zero. Meanwhile the relative uncertainty*** ε ***tents to a fixed value different from zero. Actually,*** ε ***can never be equal to zero.***