A SURVEY OF UNCERTAINITY IN PHYSICS LABS
K.Angoni, Physics Department, Vanier College

First Step   ***VERIFYING THE VALIDITY OF RECORDED DATA***

The drawing of graphs during lab measurements is practical way to estimate quickly:
   a) Whether the measurements confirm the expected behaviour predicted by physics model
   b) If any of recorded data is measured in wrong way and must be excluded from further data treatments.

Example: We drop an object from a window and we expect it to hit ground after 2sec. To verify our prediction, we measure this time several times and record the following results;
        1.99s, 2.01s, 1.89s, 2.05s 1.96s, 1.99s, 2.68s, 1.97s, 2.03s, 1.95s
   Note: ***3-5 measurements*** is a ***minimum acceptable number of data***, i.e. repeat the measurement 3-5 times.

To check out those data we include them in a graph (fig.1). From this graph we can see that:
   a) The fall time seems to be *constant* and very likely ~2s. So, in general, we have acceptable data.
   b) The seventh measure seems too far from the others results and this might be due to an abnormal circumstance during its measurement. To eliminate any doubt, we ***exclude*** this value from the following data analysis. We have enough other data to work with. Our remaining data are: 1.99s, 2.01s, 1.89s, 2.05s, 1.96s, 1.99s, 1.97s, 2.03s, 1.95s. .
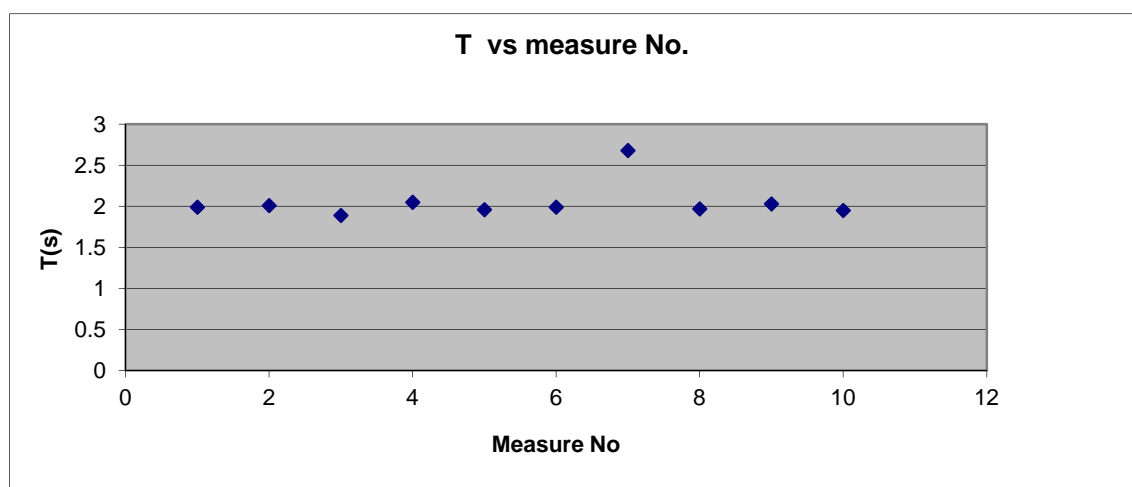

  Fig.1

Second step   ***ORGANIZING RECORDED DATA IN A TABLE***

Include all data in a table organized in such a way that some cells be ready to include the uncertainty calculation results. In our example, we are looking to estimate a single parameter "T", so we have to predict two cells for its average and its uncertainty.

Table_1

T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	Tav	ΔT
1.99s	2.01s	1.89s	2.05s	1.96s	1.99s	1.97s	2.03s	1.95s		

Third step   **CALCULATIONS OF UNCERTAINTIES**

The **true value** of parameter is unknown. We use the *recorded data* to find an **estimation** of the **true value** and the **uncertainty** of this **estimation.**

There are three particular situations for uncertainty estimations.

A] - **We measure several times a parameter and we get always the same numerical value**.
Example: We measure the length of a table three times and we get **L= 85cm** and **a little bit more or less**. This happens because the smallest unit of the meter stick is **1cm** and we **cannot be precise** about what *portion of 1cm* is the quantity "**a little bit more or less".** In such situations we use **"the half-scale rule"** i.e.**;** **the uncertainty is equal to the half of the smallest unit available used for measurement**. In our example **ΔL= ±0.5cm and the result of measurement is reported as L= (85.0 ± 0.5)cm.**

-If we use a meter stick with **smallest unit available 1mm,** we are going to have a more precise result but even in this case there is an uncertainty. Suppose that we get always the length **L= 853mm**. *Being aware that there is always a **parallax error** (eye position) on both sides reading, one may get ΔL= ± 1 or ±2mm) depending on the measurement circumstances. **In general, we will accept 1or 2 units of measurement. The result is reported L= (853 ± 1)mm .** Our **best estimation** for the table length is **853mm** and our measurements show that the **true length** is between 852 and 854mm. The **uncertainty interval** is **(852, 854)mm** and the **absolute uncertainty** of estimation is **ΔL= ± 1mm**

-Let's suppose that using the same meter stick, we measure the length of a calculator and a room and find **L$_{calc}$= (14.0 ±0.5)cm and L$_{room}$= (525.0 ±0.5)cm.** In the two cases we have the same absolute uncertainty **ΔL= ± 1m** but we are *conscious that the length of calculator is measured more precisely. The **precision of a measurement** is estimated by the uncertainty portion that belongs to the unit of*

*measurement quantity. Actually, it is estimated by the **relative error*** $\varepsilon = \dfrac{\Delta L}{\bar{L}} * 100\%$

-**Smaller relative error** means **higher precision** of measurement. In our length measurement, we have $\varepsilon_{calc} = \dfrac{0.5}{14} * 100\% = 3.57\%$ and $\varepsilon_{room} = \dfrac{0.5}{525} * 100\% = 0.095\%$. We see that the room length is measured much more precisely (about 40 times).

   Note: Don't mix the **precision** with **accuracy**. A measurement is **accurate** if **uncertainty interval** contains an *expected* (**known**) *value* and **non accurate** if it does not contain it.

B] **We measure several times a parameter and we get always different numerical values.**

Example: We drop an object from a window and we measure the time it takes to hit ground. We find the different values of time intervals inserted in table _1. In cases like this, we have to calculate the *average value* and the **absolute uncertainty** based on statistical methods.

b.1) We calculate the **best estimate** of parameter by the **average of measured** data . For our example

$$\bar{T} = \frac{1}{n}\sum_{i=1}^{n} T_i = \frac{1}{9}\sum_{i=1}^{9} T_i = \frac{1}{9}[1.99 + 2.01 + 1.89 + 2.05 + 1.96 + 1.99 + 1.97 + 2.03 + 1.95] = 1.982s$$

b.2) We estimate the **absolute uncertainty** through the **spread of recorded data**.
   A first way to calculate the absolute uncertainty is by **mean deviation** i.e. "*average distance*" of **data**

from their *average value*. In case of our example

we get
$$\Delta T = \frac{1}{n}\sum_{i=1}^{n}\left|T_i - \bar{T}\right| = \frac{1}{9}\sum_{i=1}^{9}\left|T_i - 1.982\right| = 0.035 \text{s}$$

Now we can say that the **real value** *of fall time* is inside the <u>**uncertainty interval**</u> (1.947, *2.017*)*s* or between $T_{max} = 2.017s$ and $T_{min} = 1.947s$ with **best estimation** (average value) 1.982*s*.

The result is reported as   **T = (1.982 +/- 0.035)s**

The "**standard**[1] **deviation**" of data is another *(statistically better)* estimation of <u>absolute uncertainty</u>.

Based on our example data we get
$$\sigma T = \sqrt{\frac{\sum_{i=1}^{n}\left(T_i - \bar{T}\right)^2}{n-1}} = \sqrt{\frac{\sum_{i=1}^{9}\left(T_i - 1.982\right)^2}{8}} = 0.047s.$$

The result is reported as   **T = (1.982 +/- 0.047)s**

b.3) In general terms, a *larger interval of uncertainty* means a more "*conservative estimation*" but in the same time a *more reliable estimation*. That's why the *standard deviation* is a better estimation for the *absolute uncertainty*. Note that we get $\Delta T = $ *+/- 0.047s* when using the *standard deviation* and $\Delta T = $ *+/- 0.035s* when using the *mean deviation. Also, the relative error (i.e. relative uncertainty)* calculated from the *standard deviation* is bigger. In our example the *relative uncertainty* of measurements is

$$\varepsilon = \frac{\sigma T}{\bar{T}}*100\% = \frac{0.047}{1.982}*100\% = 2.37\%$$ when using the **standard deviation**

and $$\varepsilon = \frac{\Delta T}{\bar{T}}*100\% = \frac{0.035}{1.982}*100\% = 1.81\%$$ when using the **mean deviation**

Note: We will accept that our measurement is enough precise if the *relative uncertainty* "**ε**" of measurements for the *same value* of a physical parameter is 5-10%

If the relative uncertainty is > 10%, we may proceed by:

a) *Cancelling* the data "**shifted the most from the average value**";
b) *Increasing* the number of data by repeating more times the measurement;
c) *Improving* the measurement procedure.

C] Estimation of Uncertainties for Calculated Quantities (Uncertainty propagation)

- Very often, we use the experimental data recorded for some parameters and a mathematical expression to estimate the value of a *parameter of interests* (POI). As we estimate the *measured parameters* with a certain *uncertainty*, it is clear that the estimation of POI with have an uncertainty.

---

[1] The standard deviation can be calculated direct in Excel and in many calculators.

- Actually, the calculation of POI *average* is based on the *averages of measured parameters* and the formula that relates POI with measured parameters. Meanwhile, the uncertainty of POI estimation is calculated by using the *Max_Min* method. This method calculates the limits of *uncertainty interval*, POI_{min} *and* POI_{max} by using the formula relating POI with other parameters and the combination of their limit values in such a way that the result be the smallest or the largest possible.

Example. To find the volume of a rectangular pool with constant depth , we measure its length, its width and its depth and then, we calculate the volume by using the formula V=L*W*D. Assume that our measurement results are L = (25.5 ± 0.5)m, W = 12.0 ±0.5m, D = 3.5 ±0.5m

In this case the **best estimation** for the volume is V_{best} = 25.5*12.0*3.5=**1071.0 m³**. This estimation of volume is associated by an uncertainty calculated by Max-Min methods as follows
$V_{min}=L_{min}*W_{min}*D_{min}$= 25*11.5*3 = **862.5m³** and $V_{max}=L_{max}*W_{max}*D_{max}$= 26*12.5*4 = **1300.0m³**
So, the *uncertainty interval* for volume is (862.5, 1300.0) and the *absolute uncertainty* is
ΔV= (V_{max}-V_{min})/2 = (1300.0 - 862.5)/2= 218.7m3 while the **relative error** is $\varepsilon_V = \dfrac{218.7}{1071.0}*100\% = 20.42\%$

Note_1: When applying the Max-Min method to calculate the uncertainty, one must pay attention to the mathematical expression that relates POI to measured parameters.

Examples: - You *measure* the *period* of an oscillation and you use it to *calculate* the *frequency* (POI). As $f = 1/T$, $f_{av} = 1/T_{av}$ the *max-min method* gives $f_{min}=1/T_{max}$ and $f_{max}=1/T_{min}$

- If $z = x - y$, $z_{av} = x_{av} - y_{av}$ and $z_{MAX} = x_{MAX} - y_{MIN}$ and $z_{MIN} = x_{MIN} - y_{MAX}$.

Note_2. *If the best values of parameters in expression are missing* one may use $POI_{AVERAGE}$ as the best POI estimation $POI_{AVER} = \dfrac{POI_{MAX} + POI_{MIN}}{2}$. But, if *best values of parameters in expression* are available, use them to find *best value estimation. POI$_{AV}$ is not always equal to* **POI best value estimation.** So, for the pool volume V_{AVER}= (1300+862.5)/2 =**1081.25m3** which is different from V_{best} =**1071.0 m³**

How to present the result of uncertainty calculations?
You must provide the **average,** the **absolute uncertainty** and the **relative uncertainty.** So, for the last example, the result of uncertainty calculations should be presented as follows:
V= (1071.0 ± 218.7) m³ , ε =20.4%

Note: *Uncertainties* must be *quoted* to the **same number of decimal digits** as the **average value**. The use of *scientific notation* helps to prevent confusion about the number of significant figures.
Example: If calculations generate, say   A = 0.03456789 ± 0.00245678.
This should be presented after being rounded off (leave 1,2 or 3 digits after decimal point):

$$A = (3.5 \pm 0.2) \times 10^{-2} \text{ or } A = (3.46 \pm 0.25) \times 10^{-2}$$

HOW TO CHECK WHETHER TWO QUANTITIES ARE EQUAL?

This question appears essentially in two situations:
a) We measure the *same parameter* by two *different methods* and want to verify the results are equal.
b) We use measurements to *verify* if *a theoretical expression* is right.

In the first case, we have to compare the estimations $A \pm \Delta A$ and $B \pm \Delta B$ of the "two parameters". The second case can be transformed easily to the first case by noting the left side of expression A and the right side of expression B. Then, the procedure is the same. **Example**: We want to verify if the thins lens equation $1/p + 1/q = 1/f$ is right. For this we note $1/p + 1/q = A$ and $1/f = B$

Rule: We will consider that the quantities A and B are **equal**[2] if their **uncertainty intervals overlap**.

Fig.2
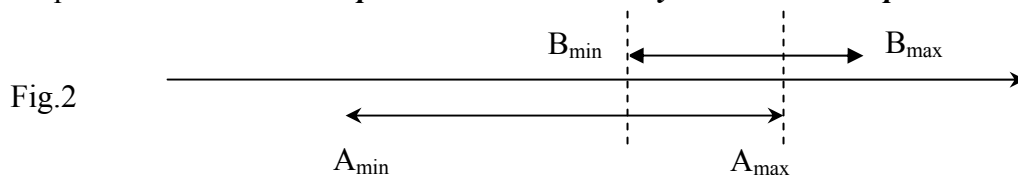
WORK WITH GRAPHS

Very often, one uses the graphs to **check the validity of a theoretical expression** or to **find the values** of physical quantities.

Example; We find theoretically that the oscillation period of a simple pendulum is $T = 2\pi\sqrt{L/g}$ and we wants to verify it experimentally. For this, as a first step, we prefer to get a linear relationship between two quantities we can measure; in our case period T and length L. For this we square the two sides of the

relation $T^2 = \dfrac{4\pi^2}{g} * L$ pose $T^2 = y$, $L = x$ and get the _linear expression_ $y = a*x$ where $a = 4\pi^2/g$.

So, we have to verify experimentally if there is such a relation between T^2 and L. **Note that if this is verified we can use the experimental value of a to calculate the free fall constant value "$g = 4\pi^2/a$".**

- Assume that, after measuring several times the period for a given pendulum length(x value), calculated the **average** and **uncertainty** for $y(T^2)$ and repeated this for a set of different values of length $x(1...6m)$, we get the data shown in table No 1. At first, we graph the average data. We see that they are aligned on a straight line, as expected. Then, we use Excel to find the best linear fitting for our data and we ask this line to pass from (x = 0, y = 0) _because this is predicted from the theoretical formula_. We get a straight line with $a_{av} = 4.065$. Using our theoretical formula we calculate the estimation for $g_{av} = 4\pi^2/a_{av} = 4\pi^2/4.065 = 9.70$ which is not far from **expected value 9.8.** Next, we _add the uncertainties in the graph and draw the best linear fitting with **maximum /minimum slope** that pass by origin. From **these graphs** we get $a_{min} = 3.635$ and $a_{max} = 4.202$. So, we get $g_{min} = 4\pi^2/a_{max} = 4\pi^2/4.202 = 9.38$ and $g_{max} = 4\pi^2/a_{min} = 4\pi^2/3.635 = 10.85$

Table_2

X	Y(av.)	ΔY (+/-)	Ymin	Ymax	Max. Slope	Min. Slope
1	4	1.5	2.5	5.5	4.202	3.635
2	8.3	1.8	6.5	10.1		
3	11.8	1.3	10.5	13.1	P1 (1; 1.5)	P1 (1; 5.5)
4	17	1.6	15.4	18.6	P2 (6; 25.5)	P2 (6; 21.5)
5	21	1.1	19.9	22.1		
6	23.5	2	21.5	25.5		

This way, by using the graphs we:
1- have **proved experimentally** that **our relation** between T and L is right.
2- find that our measurements are **accurate** because the **uncertainty interval** (9.38, 10.85) for "**g**" does _include the officially accepted value g = 9.8m/s²_
3-find the **absolute error** $\Delta g = (10.85-9.38)/2 = 0.735 m/s^2$
The relative error is $\varepsilon = (0.735/9.70)*100\% = 7.6\%$ which means a acceptable ($< 10\%$) **precision of measurement**.

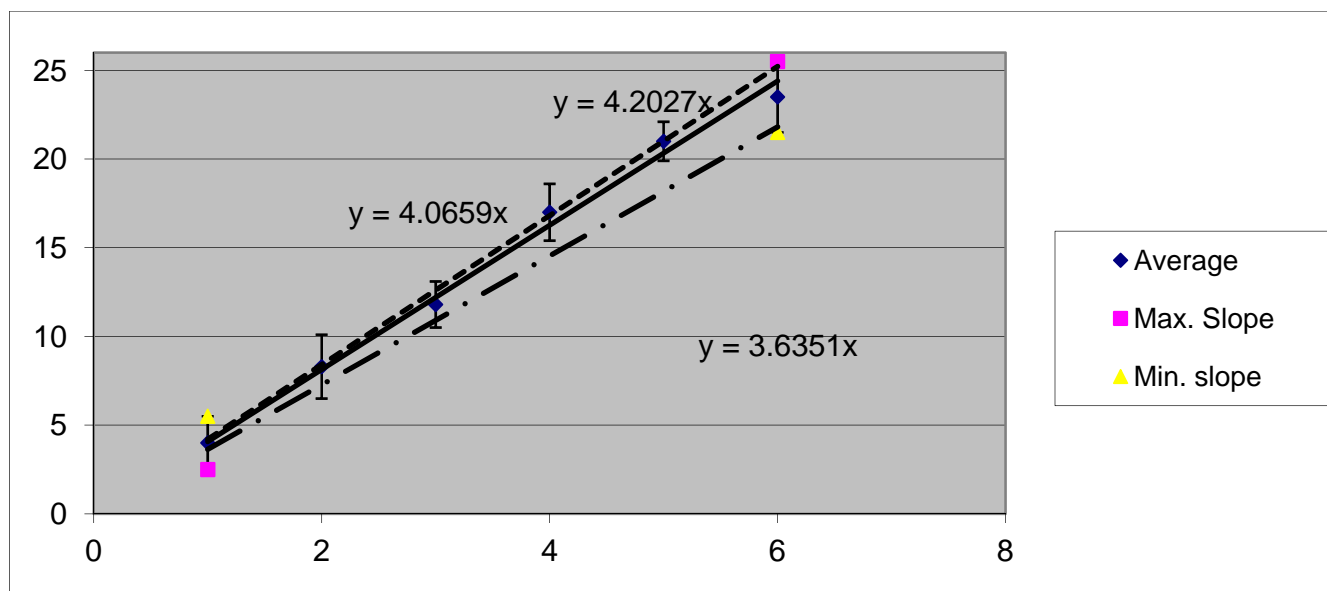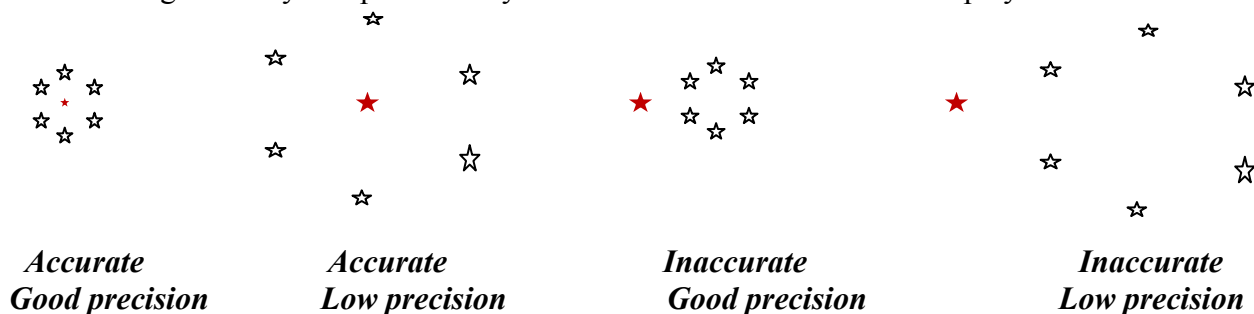[2] They should be expressed in the same unit, for sure.

Fig.3

ABOUT THE ACCURACY AND PRECISION

- Understanding accuracy and precision by use of hits distribution in a Dart's play.

| **Accurate** | **Accurate** | **Inaccurate** | **Inaccurate** |
| **Good precision** | **Low precision** | **Good precision** | **Low precision** |

- As a rule, before using a method (or device) for measurements, one should verify that the method produces **accurate results** in the range of expected values for the parameter under study. This is an obligatory step in research and industry and it is widely known as the calibration procedure. During a calibration procedure one records a set of data and makes sure that the **result is accurate**.

In principle, the *result of experiment is accurate* if the "*average of data*" fits to the" *officially accepted value*". We will consider that our experiment is "*enough accurate*" if the" *officially accepted value*" falls **_inside_** the interval of uncertainty of measured parameter; *otherwise the result is inaccurate.*

The quantity $\varepsilon_{accu} = \dfrac{\left| C_{Av} - C_{off} \right|}{C_{off}} x100$ % *(often ambiguously named as **error**)* gives the **relative shift of**

average from the officially accepted value $C_{official}$. It is clear that **the accuracy is higher** when ε_{accu} is **smaller. But**, the measurement is **_inaccurate if_** $\varepsilon_{accu} > \varepsilon$ *(relative uncertainty of measurement).*

Remember that relative uncertainty $\varepsilon = \dfrac{\Delta C}{C_{Av}} x100$ % **is different from ε_{accu}.**

Note: For an *accurate measurement* and for *a big number of measured data*, the *average* should fit to the *expected value* of parameter and *ε_{accu} should be practically zero.* Meanwhile ε tents to a fixed value different from zero. Actually, ε can never be equal to zero.

CALCULATION OF ERROR PROPAGATION IN SIMPLE EXPRESSIONS.

Assume that one knows $x_{best} \pm \Delta x$ *(say 5.2, ± 0.5)* ; $y_{best} \pm \Delta y$ *(say 11.25, ± 0.05)*, $z_{best} \pm \Delta z$ *(say 0.8 ± 0.1).*

a) a.1 If **C = a*x** where a is a constant *then* $C_{best} = a*x_{best}$ and $\Delta C = a* \Delta x$
   a.2 If **C = a/x^n** where a is a constant *then* $lnC = lna - n*lnx$ and $\Delta C/C_{best} = 0 + n*\Delta x/x_{best}$
   This way $\varepsilon_C = n*\varepsilon_x$ and $\Delta C = C_{best*} \varepsilon_c/100$ and finally $C = C_{best} \pm \Delta C$
   Ex_1: For a =2; C_{best} =2* 5.2 =10.4 ΔC =2* 0.5= 1.0 and C = 10.4 ± 1.0
   Ex_2: For a =2 and n=2; ε_x = 0.5/5.2*100 = 9.62%; ε_C = 2*9.62=19.2; C_{best} =2/ 5.2^2 =0.074
   ΔC = 0.074*19.2/100=0.014 and C = 0.07 ±0.01

b) If **C = x ± y ± z** *then* $C_{best} = x_{best} \pm y_{best} \pm z_{best}$ *and* $\Delta C = \Delta x + \Delta y + \Delta z$. So $C = C_{best} \pm \Delta C$
   Ex: C_{best} = 5.2 +11.25-0.8 =15.6 ΔC = 0.5 + 0.05 + 0.1= 0.6 and C = 15.6±0.6

c) If **C = x /y** *then* $C_{best} = x_{best} /y_{best}$ As $lnC = lnx - ln y$ and $\Delta C/C_{best} = \Delta x/x_{best} + \Delta y/y_{best}$ one gets
   $\Delta C = C_{best}(\Delta x/x_{best} + \Delta y/y_{best})$ and $C = C_{best} \pm \Delta C$. Note that $\varepsilon_C = \varepsilon_x + \varepsilon_y$ and in old textbooks, one uses
   often the notation **C = (x_{best} ± ε_x)/(y_{best} ± ε_x)** which means that one should at first calculate as $\varepsilon_C = \varepsilon_x + \varepsilon_y$
   then $\Delta C = C_{best*} \varepsilon_c/100$ and finally $C = C_{best} \pm \Delta C$.
   Ex: C_{best}=5.2 /11.25=0.46; ε_x = 0.5/5.2*100 = 9.62%; ε_y = 0.05/11.25*100=0.44%; ε_C =9.62+0.44 =10.06 %
   ΔC = 0.46$_*$ ε_c/100 = 0.46*10.06/100 = 0.05 and C = 0.46 ±0.05

d) If **C = x *y** *then* $C_{best} = x_{best} *y_{best}$ As $lnC = lnx + ln y$ and $\Delta C/C_{best} = \Delta x/x_{best} + \Delta y/y_{best}$ one gets
   $\Delta C = C_{best}(\Delta x/x_{best} + \Delta y/y_{best})$ and $C = C_{best} \pm \Delta C$. Sot $\varepsilon_C = \varepsilon_x + \varepsilon_y$ and **C = (x_{best} ± ε_x)*(y_{best} ± ε_x)** which
   means that one should at first calculate $\varepsilon_C = \varepsilon_x + \varepsilon_y$ then $\Delta C = C_{best*} \varepsilon_c/100$ and finally $C = C_{best} \pm \Delta C$.
   Ex: C_{best}= 5.2 *11.25=58; ε_x = 0.5/5.2*100=9.62%; ε_y =0.05/11.25*100=0.44%; ε_C=9.62+ 0.44=10.06%
   ΔC = 58.4$_*$ ε_c/100 = 58.4*10.06/100= 5.8 and C = 58 ±6

Remember: The result of a calculation cannot have higher precision than any of terms.
So, one starts by completing the mathematical calculations and follows by keeping at the result the number of digits that fits to the less precise term (*conservator principle*). In practice, one has to note the *significant figure* and the *number of decimals* for each term. Then, depending on the form of mathematical expression, one applies the rules of Sig.Fig. and DEC to identify the uncertain digit at the result. Finally, one rounds off the result.
In college level, we will prefer to keep high values of absolute uncertainty justifying this by conservator principle. Meanwhile, for *realistic estimations* one should tent to refer to the smallest reasonable value for absolute uncertainty. This is done by use of some particular math techniques taught at university level.

CALCULATION OF UNCERTAINTY PROPAGATION BY USE OF DIFFERENTIALS

-The derivative of a function *y = y(x)* is noted   ***y'(x) = dy/dx   (1)***
In this expression, the ***differentials dx*** and ***dy*** represent the infinitesimal small change of quantities *x, y*.
Based on expression (1) we may relate these differentials   ***dy = y'(x)* dx   (2)***
*The **mathematical differentials** are extremely small and **non measurable** but if we assume that the derivative of function **y'** remains almost constant **in a small but measurable region Δx of x- values, we can write the relation (2) in the form**   **Δy = y'(x)*Δx   (3)**
This relation is used very efficiently in physics for error propagation calculation (type 3).
If the function has two variables *x₁, x₂,* **the expression (3) becomes   Δy = y'ₓ₁*Δx₁ + y'ₓ₂*Δx₂ (4)**

-Ex_1. Measuring the length of an object by using a meter stick; **smallest unit is 1cm**. The procedure consist in reading the positions x_1, x_2 of two object ends and calculating its length as $L = x_2 - x_1$
In this case we get *0.5cm* absolute *uncertainty* during the *reading process*. So, if we read $x_1 = $ **56cm** we have $\Delta x_1 = $ *0.5cm and if we* **read** $x_2 = $ 96cm *we have $\Delta x_2 = 0.5cm$, too.* Then, we calculate the length of object as $L = 96 - 56 = 40cm.$ To get the **absolute uncertainty of L-value** we use eq.**(4).** It comes out that $L'_{x1} = -1$, $L'_{x2} = 1$ and $\Delta L = +1*\Delta x_2 - 1*\Delta x_1 = \Delta x_2 - \Delta x_1$
Before proceeding with numerical calculation we substitute "-" by "+" because we wants to refer to the **worst case of uncertainty.** *This is known as the* **conservative approach** *in* **uncertainty calculations.**
So, we get $\Delta L = \Delta x_2 + \Delta x_1 = 0.5 + 0.5cm = 1cm$ **and** $\varepsilon = \Delta L/L*100\% = 1/40*100 = 2.4\%$

-Ex_2. We measure the period of an oscillation and find $T = 5.5 \pm 0.5s$.
Meanwhile, for the purposes of the experiment, we need to calculate an estimation for the quantity $y = T^2$
In this case, we may proceed quickly by using the derivative $y' = dy/dT = 2T$ and $dy = 2TdT$. Then, in physics, $\Delta y = 2T\Delta T$. Finally $y = y_{Av} \pm \Delta y = 5.5^2 \pm 2*5.5*0.5 = (30.25 \pm 5.5)$ [s^2]

The max-min method requires a special attention to the form of mathematical expression. If the considered expression contains many variables, often one prefers to use the differential method. The two following examples make easier to understand some advantages of differential method.

Ex_3. The measurements results for three physical parameters are **A= 5.1 ± 0.3; B = 25 ± 1;**
    C = 3.45 ± 0.05. Calculate the **average, maximum, minimum** values and **relative**
    uncertainty for $y = A * B - \frac{B*C^2}{A}$. **Pay attention to sig. figures and rounding off rules.**

    3-a) **Differential method** *(one starts by mathematical differential)*

- Calculate the mathematical differential of function y;
$$dy = dA * B + AdB - dB\frac{C^2}{A} - \frac{B*2C*dC}{A} - B*C^2(-1)A^{-2}dA$$

- Convert *mathematical differentials* to *physical differentials* i.e. **physical uncertainties** and apply the **conservative principle** *(* everywhere + sign)
$$\Delta y = \Delta A * B + A * \Delta B + \frac{C^2}{A} * \Delta B + \frac{2BC}{A} * \Delta C + \frac{BC^2}{A^2} * \Delta A$$

- Substitute the known *best* and *absolute uncertainty* values for A , B and C into this expression
$$\Delta y = 0.3 * 25 + 5.1 * 1 + \frac{3.45^2}{5.1} * 1 + \frac{2*25*3.45}{5.1} * 0.05 + \frac{25*3.45^2}{5.1^2} * 0.3$$
$$\Delta y = \frac{7.5}{1SF = 0Dec} + \frac{5.1}{1SF = 0Dec} + \frac{2.3338}{1SF = 0DEC} + \frac{1.691}{1SF = 0DecC} + \frac{3.432}{1SF = 0DecC} = \frac{20.0568}{0Dec = 1SF} \equiv \mathbf{20}$$

Also, $$y_{Best} = 5.1 * 25 - \frac{25*3.45^2}{5.1} = \frac{127.5}{2SF = 0Dec} - \frac{58.345}{2SF = 0Dec} = \frac{69.15}{0Dec} = \mathbf{69}$$

$$y = y_{Best} \pm \Delta y = 69 \mp 20;\ y_{Min} = y_{Av} - \Delta y = 69 - 20 = \mathbf{49};\ y_{Max} = y_{Av} + \Delta y = 69 + 20 = \mathbf{89}$$

And, the relative uncertainty is $$\epsilon = \frac{\Delta y}{y_{Av}} * 100\% = \frac{20}{69} * 100 = \mathbf{29\%} \equiv \mathbf{30\%}$$

    3-b) **Max-Min method** (One must pay special attention to the form of math. expression)

$$y_{Best} = 5.1 * 25 - \frac{25 * 3.45^2}{5.1} = \frac{127.5}{2SF = 0Dec} - \frac{58.345}{2SF = 0Dec} = \frac{69.15}{0Dec} = \mathbf{69}$$

$$y_{Min} = A_{Min} * B_{Min} - \frac{B_{Max} * C_{Max}^2}{A_{Min}} = 4.8 * 24 - \frac{26 * 3.5^2}{4.8} = \frac{115.2}{2SF = 0Dec} - \frac{66.3541}{2SF = 0Dec} = \frac{48.845}{0Dec = 1SF} \equiv \mathbf{49}$$

$$y_{Max} = A_{Max} * B_{Max} - \frac{B_{Min} * C_{Min}^2}{A_{Max}} = 5.4 * 26 - \frac{24 * 3.4^2}{5.4} = \frac{140.4}{2SF = 0Dec} - \frac{51.3777}{2SF = 0Dec} = \frac{89.022}{0Dec = 1SF} \equiv \mathbf{89}$$

$$\Delta y = (y_{Max} - y_{Min})/2 = (89\text{-}49)/2 = \mathbf{20} \quad \text{and} \quad \epsilon = \frac{\Delta y}{y_{Av}} * 100\% = \frac{20}{69} * 100 = \mathbf{29\%} \equiv \mathbf{30\%}$$

Ex_4. The data for three physical parameters are A= **5.1 ± 0.3; B = 25 ± 1; C = 3.45 ± 0.05.**

Calculate the **average, maximum, minimum** values and **relative uncertainty** for $y = \frac{A^2 * C^5}{B^3}$.

*In this case the expression contains **only** the **product** and **the power** of parameters.*

4a) Differential method starting by natural logarithms

-Calculate the best value $\quad \mathbf{y_{Best}} = \frac{A_{Best}^2 * C_{Best}^5}{B_{Best}^3} = \frac{5.1^2 * 3.45^5}{25^3} = \frac{26.1 * 488.7597966}{15625} = \frac{0.81642436}{2SF} \equiv \frac{\mathbf{0.82}}{2SF}$

- Take the natural logarithm of expression $\quad lny = lnA^2 + lnC^5 - lnB^3 = 2lnA + 5lnC - 3lnB$

- Take the differential of both sides $\quad\quad\quad\quad\quad \frac{dy}{y} = 2\frac{dA}{A} + 5\frac{dC}{C} - 3\frac{dB}{B}$

- Convert *mathematical differentials* to *physical differentials* i.e. **physical uncertainties** and apply the
conservative principle *(*put everywhere + sign) $\quad\quad \frac{\Delta y}{y} = 2\frac{\Delta A}{A} + 5\frac{\Delta C}{C} + 3\frac{\Delta B}{B}$

$$\frac{\Delta y}{y} = 2\frac{0.3}{5.1} + 5\frac{0.05}{3.45} + 3\frac{1}{25} = \frac{0.117647}{1SF = 1Dec} + \frac{0.07246}{1SF = 5Dec} + \frac{0.12}{1SF = 1ec} = \frac{0.310107}{2Dec} \equiv \mathbf{0.3}$$

- Calculate the *absolute uncertainty* as $\quad\quad\quad \mathbf{\Delta y} = y_{Av} * 0.3 = 0.82 * 0.31 = \frac{0.254}{2SF} \equiv \mathbf{0.25}$

So, the result is $\quad\quad\quad\quad \mathbf{y = 0.82 \pm 0.25} \quad \textbf{and} \; \; \varepsilon = \frac{0.25}{0.82} * \mathbf{100\%} = \frac{\mathbf{0.3048}}{\mathbf{2SF}} \equiv \mathbf{30\%}$

4b) Max-Min method

$$\mathbf{y_{Best}} = \frac{A_{Best}^2 * C_{Best}^5}{B_{Best}^3} = \frac{5.1^2 * 3.45^5}{25^3} = \frac{26.1 * 488.7597966}{15625} = \frac{0.81642436}{2SF} \equiv \frac{\mathbf{0.82}}{2SF}$$

$$\mathbf{y_{Max}} = \frac{A_{Max}^2 * C_{Max}^5}{B_{Min}^3} = \frac{5.4^2 * 3.5^5}{24^3} = \frac{29.16 * 525.21875}{13824} = \frac{1.1078833}{2SF} \equiv \frac{\mathbf{1.1}}{2SF}$$

$$\mathbf{y_{Min}} = \frac{A_{Min}^2 * C_{Min}^5}{B_{Max}^3} = \frac{4.8^2 * 3.45^5}{26^3} = \frac{23.04 * 454.35424}{17576} = \frac{0.595603}{2SF} \equiv \frac{\mathbf{0.60}}{2SF}$$

$$\Delta y = (y_{Max} - y_{Min})/2 = (1.1 \text{ - } 0.6) / 2 = \mathbf{0.25} \quad \text{and} \quad \epsilon = \frac{\Delta y}{y_{Av}} * 100\% = \frac{0.25}{0.82} * 100 = \mathbf{30\%}.$$