BRIEF SURVEY OF UNCERTAINITY IN PHYSICS LABS

First Step ***VERIFYING THE VALIDITY OF RECORDED DATA***

The drawing of graphs during lab measurements is practical way to estimate quickly:

a) Whether the measurements confirm the expected behaviour predicted by physics model
b) If any of recorded data is measured in wrong way and must be excluded from further data treatments.

Example_1: We drop an object from a window and we expect it to hit ground after 2sec. To verify our prediction, we measure this time several times and record the following results;

1.99s, 2.01s, 1.89s, 2.05s 1.96s, 1.99s, 2.68s, 1.97s, 2.03s, 1.95s

(Note: ***3-5 measurements*** is a ***minimum acceptable number of data*** for estimating a parameter, i.e. repeat the measurement 3-5 times)

To check out those data we include them in a graph (fig.1). From this graph we can see that:

a) The fall time seems to be *constant* and very likely ~2s. So, in general, we have acceptable data.
b) The seventh measure seems too far from the others results and this might be due to an abnormal circumstance during its measurement. To eliminate any doubt, we ***exclude*** this value from the following data analysis. We have enough other data to work with. Our remaining data are: 1.99s, 2.01s, 1.89s, 2.05s, 1.96s, 1.99s, 1.97s, 2.03s, 1.95s. .

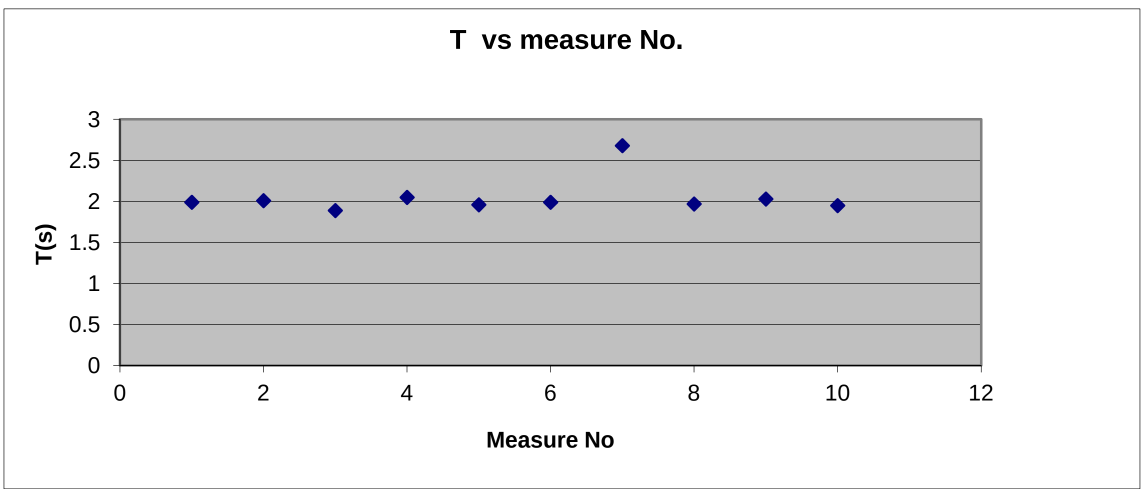


Fig.1

Second step ***ORGANIZING RECORDED DATA IN A TABLE***

Include all data in a table organized in such a way that some cells be ready to include the uncertainty calculation results. In our example, we are looking to estimate a single parameter "T", so we have to predict (*at least*) two cells for its average and its uncertainty.

Table_1

T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	Tav	ΔT
1.99s	2.01s	1.89s	2.05s	1.96s	1.99s	1.97s	2.03s	1.95s		

<u>Third step</u> ***CALCULATIONS OF UNCERTAINTIES***

The ***true value*** of parameter is unknown. We use the *recorded data* to find an ***<u>estimation</u>*** of the ***true value*** and the **<u>uncertainty</u>** of this **estimation.**

***<u>There are three particular situations for uncertainty estimations</u>*.**

A] - We measure several times a parameter and we get always the same numerical value.
Example_2: We measure the length of a table three times and we get ***L= 85cm*** and ***a little bit more or less***. This happens because the smallest unit of the meter stick is **1cm** and we ***<u>cannot be precise</u>*** about what *portion of 1cm* is the quantity "***a little bit more or less".*** In such situations we use ***"the half-scale rule"*** i.e.***; <u>the uncertainty is equal to the half of the smallest unit available used for measurement</u>***.
In our example ***ΔL= ±0.5cm and the result of measurement is reported as L= (85.0 ± 0.5)cm.***

-If we use a meter stick with **smallest unit available 1mm,** *we are going to have a more precise result but even in this case there is an uncertainty. Suppose that we get always the length* **L= 853mm***. Being aware that there is always a* **parallax error** *(eye position) on both sides reading, one may get* ***ΔL= ±0.5, ±1*** *and* ***even ±2mm****) depending on the measurement circumstances.* ***The result of measurement is reported as L= (853.0 ±0.5)mm*** *or* ***(853 ± 1)mm*** *or* ***(853 ± 2)mm .*** *Our* ***<u>best estimation</u>*** *for the table length is* ***853mm****. Also, our measurements show that the* ***true length*** *is between 852 and 854mm.* If the **<u>absolute uncertainty</u>** of estimation is ***ΔL= ± 1mm****, than* the ***<u>uncertainty interval</u>*** is ***(852, 854)mm***.

-Let's suppose that using the same meter stick, we measure the length of a calculator and a room and find ***L_{calc}= (14.0 ±0.5)cm*** *and* ***L_{room}= (525.0 ±0.5)cm.*** In the two cases we have the same absolute uncertainty ***ΔL= ± 0.5cm*** but we are *conscious that the length of room is measured more precisely. The* ***precision of a measurement*** *is estimated by the uncertainty portion that belongs to the unit of measurement quantity. Actually, it is estimated by the* ***relative error***

$$\varepsilon = \frac{\Delta L}{\bar{L}} * 100\% \qquad (1)$$

-Note that ***smaller relative error*** means ***higher precision*** of measurement. In our length measurement, we have $\varepsilon_{calc} = \frac{0.5}{14} * 100\% = 3.57\%$ and $\varepsilon_{room} = \frac{0.5}{525} * 100\% = 0.095\%$. We see that the room length is measured much more precisely (about 38 times).

Note: Don't mix the ***precision*** with ***accuracy***. A measurement is ***accurate*** if ***uncertainty interval*** contains an *expected* (**known by literature**) *value* and ***non accurate*** if it does not contain it.

B] We <u>measure</u> several times a parameter and we get always different numerical values.

Example: For data collected *in experiment_1* we have to calculate the average ***value*** and the ***absolute uncertainty*** based on statistical methods.

b.1) We *estimate* the value of parameter by the **<u>average of measured</u>** <u>data</u> . In case of our first example

$$\bar{T} = \frac{1}{n}\sum_{i=1}^{n} T_i = \frac{1}{9}\sum_{i=1}^{9} T_i = \frac{1}{9}[1.99 + 2.01 + 1.89 + 2.05 + 1.96 + 1.99 + 1.97 + 2.03 + 1.95] = 1.982s \qquad (2)$$

b.2) To estimate ***how far from the average*** can be the ***true value*** we use the ***spread of measured data***.
<u>A first way to estimate the spread</u> is by use of ***<u>mean deviation</u>*** i.e. "*average distance*" of ***data from*** their ***average value***. In case of our example

we get $$\Delta T = \frac{1}{n}\sum_{i=1}^{n}\left|T_i - \bar{T}\right| = \frac{1}{9}\sum_{i=1}^{9}\left|T_i - 1.982\right| = 0.0353\text{s} \qquad (3)$$

Now we can say that the **true value** *of fall time* is inside the ***uncertainty interval*** (1.947, *2.017*)***sec*** or between T_{max} = 2.017***s*** and T_{min} =1.947***s*** with average value 1.982***s***. Taking in account the rules on significant figures and rounding off we get T_{Av}= *1.98sec* and *ΔT= 0.04sec* and

The result is reported as ***T = (1.98 +/- 0.04)sec*** (4)

*Another (**statistically better**) estimation of spread* is the "**standard[1] deviation**" of data.

Based on our example data we get $$\sigma T = \sqrt{\frac{\sum_{i=1}^{n}\left(T_i - \bar{T}\right)^2}{n-1}} = \sqrt{\frac{\sum_{i=1}^{9}\left(T_i - 1.982\right)^2}{8}} = 0.047s\,. \qquad (5)$$

The result is reported as ***T = (1.98 +/- 0.05)sec*** (6)

b.3) For *spread estimation*, a ***larger interval of uncertainty*** means a more "***conservative estimation***" but in the same time a ***more reliable estimation***. That's why the ***standard deviation*** is a better estimation for the ***absolute uncertainty***. Note that we get ΔT= ***+/- 0.05s*** when using the ***standard deviation*** and ΔT= ***+/- 0.04s*** when using the ***mean deviation.*** *Also, the **relative error (or relative uncertainty)** calculated from the **standard deviation*** is bigger. In our example the ***relative uncertainty*** of measurements is

$$\varepsilon = \frac{\sigma T}{\bar{T}} * 100\% = \frac{0.047}{1.982} * 100\% = 2.37\%$$ when using the ***standard deviation***

and $$\varepsilon = \frac{\Delta T}{\bar{T}} * 100\% = \frac{0.035}{1.982} * 100\% = 1.81\%$$ when using the ***mean deviation***

Important: The absolute and relative uncertainty can never be zero.

Assume that you **repeat** 5 times a given measurement and you read all times the same value ***X***. So, by applying the rules of case "b" you may rapport ***X_{Av} = 5X/5= X and ΔX_b= 0. But here you*** *deal with a* ***case "a"*** and this means that there is a ***$\Delta X_a(\neq 0)$ = ½(smallest unit of measurement scale).*** This example shows that, when calculating the absolute uncertainty, *one should take into account the precise expression*

$$\Delta X = \Delta X_a + \Delta X_b \qquad (7)$$

Note that in those cases where ***$\Delta X_b \gg \Delta X_a$ one may simply disregards ΔX_a.***

Exemple: In exemple_1 the time is measured with 2 decimals. This means that ***ΔX_a= 1/2(0.01) = 0.005s*** Meanwhile (*from 6) ΔX_b= 0.05s which is **ten times bigger** than ΔX_a. In this case one **may** neglect **ΔX_a**. **But if** ΔX_b were 0.02s and ΔX_a= 0.005s one cannot neglect ΔX_a= 0.05s because it is 25% of ΔX_b. In this case one must use the expression (7) to calculate the absolute uncertainty and ΔX= 0.02 +0.005=0.025s*

Note: You will consider that a measurement has a *good precision* if the ***relative uncertainty* ε <10%** .

If the relative uncertainty is **ε** > 10%, you may proceed by:

a) *Cancelling* any particular data "***shifted too much from the average value***" ;
b) *Increasing* the number of data by repeating more times the measurement;
c) *Improving* the measurement procedure.

[1] The standard deviation can be calculated direct in Excel and in many calculators.

C] Estimation of Uncertainties for Calculated Quantities (Uncertainty propagation)

Very often, we use the experimental data recorded for some parameters and a mathematical expression to estimate the value of a given parameter of interests (POI). As we estimate the ***measured parameters*** with a certain ***uncertainty***, it is clear that the estimation of POI with have some uncertainty, too.
Actually, the calculation of ***best estimation*** for POI is based on the ***best estimations of measured parameters*** and the formula that relates POI with measured parameters. Meanwhile, the uncertainty of POI estimation is calculated by using the ***Max_Min*** method. This method calculates the limits of uncertainty interval, POI_{min} ***and*** POI_{max} by using the formula relating POI with other parameters and the combination of their limiting values in such a way that the result be the smallest or the largest possible.

Example. To find the volume of a rectangular pool with constant depth , we measure its length, its width and its depth and then, we calculate the volume by using the formula V=L*W*D. Assume that our measurement results are L = (25.5 ± 0.5)m, W = 12.0 ±0.5m, D = 3.5 ±0.5m

In this case the **best estimation** for the volume is V_{best} = 25.5*12.0*3.5=**1071.0 m^3**. This estimation of volume is associated by an uncertainty calculated by Max-Min methods as follows

$V_{min}=L_{min}*W_{min}*D_{min}$= 25*11.5*3 = 862.5m^3 and $V_{max}=L_{max}*W_{max}*D_{max}$= 26*12.5*4 = 1300.0m^3

So, the ***uncertainty interval*** for volume is (862.5, 1300.0) and the ***absolute uncertainty** is*

$\Delta V= (V_{max}-V_{min})/2$ = (1300.0 - 862.5)/2= 218.7m3 while the **relative error** is $\varepsilon_V = \frac{218.7}{1071.0}*100\% = 20.42\%$

Note_1: When applying the Max-Min method to calculate the uncertainty, one must pay attention to the mathematical expression that relates POI to measured parameters.

Examples: - You ***measure*** the ***period*** of an oscillation and you use it to ***calculate*** the ***frequency*** (POI). As $f = 1/T$, $f_{av} = 1/T_{av}$ *the* ***max-min method*** *gives* $f_{min}=1/T_{max}$ ***and*** $f_{max}=1/T_{min}$

- *If* $z = x - y$, $z_{av} = x_{av} - y_{av}$ *and* $z_{MAX} = x_{MAX} - y_{MIN}$ *and* $z_{MIN} = x_{MIN} - y_{MAX}$.

Note_2. Use the *best estimations of parameters in the expression* to calculate the *best estimation for POI. If they are missing* one may use POI_{middle} as the best estimation for POI

$$POI_{middle} = \frac{POI_{MAX} + POI_{MIN}}{2} \qquad (8)$$

Be aware though, that POI_{middle} is not always equal to **POI *best estimation.***
So, for the pool volume **V_{middle}**= (1300+862.5)/2 =**1081.25m3** which is different from **V_{best} =1071.0 m^3**

***<u>How to present the result of uncertainty calculations</u>*?** You must provide the **best estimation,** the **absolute uncertainty** and the **relative uncertainty.** So, for the last example, the result of uncertainty calculations should be presented as follows: **V= (1071.0 ± 218.7) m^3 , ε =(218/1071)*100%= 20.42%**

Note: ***Uncertainties*** must be ***quoted*** to the ***<u>same number of decimal digits</u> as*** the **<u>best estimation</u>**. The use of *scientific notation* helps to prevent confusion about the number of significant figures.

Example: If calculations generate, say A = (0.03456789 ± 0.00245678.)m
This should be presented after being rounded off (leave 1,2 or 3 digits after decimal point):

A = (3.5 ± 0.2) * 10^{-2}m or A = (3.46 ± 0.25) * 10^{-2}m

HOW TO CHECK WHETHER TWO QUANTITIES ARE EQUAL?

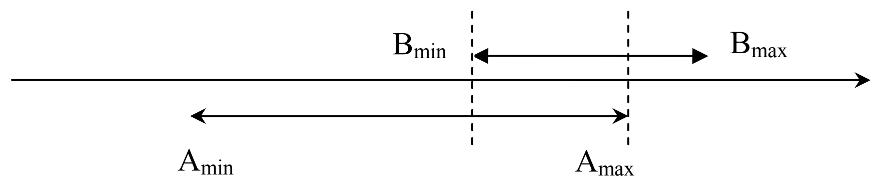
This question appears essentially in two situations:

1. We measure the ***same parameter*** by two ***different methods*** and want to verify if the results are equal.
2. We use measurements to ***verify*** if ***a theoretical expression*** is right.

In the first case, we have to compare the estimations $A \pm \Delta A$ and $B \pm \Delta B$ of the "two parameters". The second case can be transformed easily to the first case by noting the left side of expression ***A*** and the right side of expression ***B***. Then, the procedure is the same. ***Example***: We want to verify if the thins lens equation ***1/p + 1/q = 1/f*** is right. For this we note ***1/p + 1/q = A*** and ***1/f = B***

Rule: We will consider that the quantities ***A*** and ***B*** are ***equal***[2] if their ***uncertainty intervals overlap***.

Fig.2



WORK WITH GRAPHS

We use graphs to ***check the theoretical expressions*** or to ***find the values*** of physical quantities.

Example; We find theoretically that the oscillation period of a simple pendulum is $T = 2\pi\sqrt{L/g}$ and we wants to verify it experimentally. For this, as a first step, we prefer to get a linear relationship between two quantities we can measure; in our case period T and length L. For this we square the two sides of the relation $T^2 = \frac{4\pi^2}{g} * L$ pose $\boldsymbol{T^2 = y}$, $\boldsymbol{L = x}$ and get the <u>linear expression</u> $\boldsymbol{y = a*x}$ where $\boldsymbol{a = 4\pi^2/g}$.

So, we have to verify experimentally if there is such a relation between $\boldsymbol{T^2}$ and **L. *Note that if this is verified we can use the experimental value of a to calculate the free fall constant value "g = 4π²/a".***

- Assume that after measuring the period for a given pendulum length several times, calculated the ***average values*** and ***uncertainties*** for y***(=T²)*** and repeated this for a set of different values of length ***x(L=1,...,6m)***, we get the data shown in table No 1. At first, we graph the average data. We see that they are aligned on a straight line, as expected. Then, we use Excel to find the best linear fitting for our data and we ask this line to pass from (x = 0, y = 0) *because this is predicted from the theoretical formula*. We get a straight line with $\boldsymbol{a_{av} = 4.065}$. Using our theoretical formula we calculate the estimation for $\boldsymbol{g_{av} = 4\pi^2/a_{av} = 4\pi^2/4.065 = 9.70}$ *which is not far from* ***expected value 9.8.*** Next**,** we *add the uncertainties in the graph and draw the best linear fitting with* ***maximum /minimum slope*** *that pass by origin.* ***From the graphs*** *we get* $a_{min} = 3.635$ and $a_{max} = 4.202$. So, we get $\boldsymbol{g_{min} = 4\pi^2/a_{max} = 4\pi^2/4.202 = 9.38}$ ***and*** $\boldsymbol{g_{max} = 4\pi^2/a_{min} = 4\pi^2/3.635 = 10.85}$

Table_2

X	Y(av.)	ΔY (+/-)	Ymin	Ymax	Max. Slope	Min. Slope
1	4	1.5	2.5	5.5	4.202	3.635
2	8.3	1.8	6.5	10.1		
3	11.8	1.3	10.5	13.1	P1 (1; 1.5)	P1 (1; 5.5)
4	17	1.6	15.4	18.6	P2 (6; 25.5)	P2 (6; 21.5)
5	21	1.1	19.9	22.1		
6	23.5	2	21.5	25.5		

This way, by using the graphs we:

1- have ***proved experimentally*** that ***our relation*** between ***T*** and ***L*** is right.

2- find out that our measurements are ***accurate*** because the ***uncertainty interval*** (9.38, 10.85) for "**g**" does *include the officially accepted value* $\boldsymbol{g = 9.8 m/s^2}$

3-find the ***absolute error*** $\Delta g = (10.85-9.38)/2 = 0.735 m/s^2$

The relative error is $\varepsilon = (0.735/9.70)*100\% = 7.6\%$ which means a acceptable (< 10%) ***precision of measurement***.

[2] They should be expressed in the same unit, for sure.

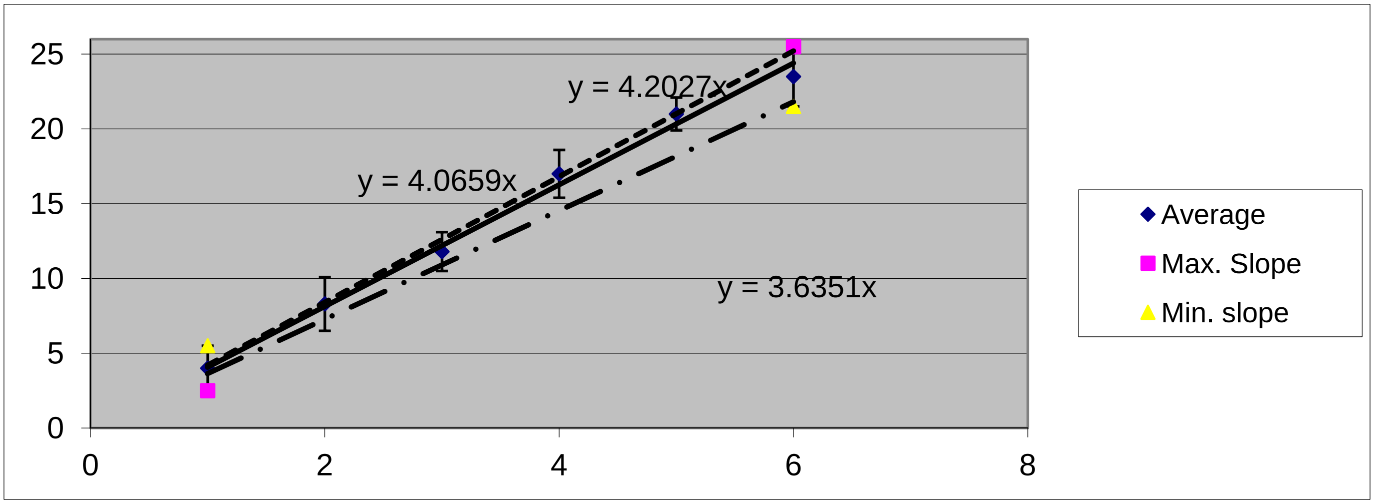


Fig.3

ABOUT THE ACCURACY AND PRECISION

- Understanding **accuracy** and **precision** by use of hits distribution in a Dart's play.

Accurate ***Good precision***	***Accurate*** ***Low precision***	***Inaccurate*** ***Good precision***	***Inaccurate*** ***Low precision***

- As a rule, before using a method (or device) for measurements, one should verify that the method produces ***accurate results*** in the range of expected values for the parameter under study. This is an obligatory step in research and industry and it is widely known as the calibration procedure. During a calibration procedure one records a set of data and makes sure that the ***result is accurate***.

In principle, the *result of experiment is accurate* if the "*average of data*" fits to the" *officially accepted value*". We will consider that our experiment is "*enough accurate*" if the" *officially accepted value*" falls ***inside*** *the interval of uncertainty* of measured parameter; *otherwise the result is inaccurate.*

The quantity $\varepsilon_{accu} = \frac{|C_{Av} - C_{off}|}{C_{off}} x100 \quad \%$ *(often ambiguously named as* ***error***) gives the ***relative shift of average from the officially accepted value*** $C_{official}$. *It is clear that* ***the accuracy is higher*** *when* ε_{accu} ***is smaller. But****, the measurement is* ***inaccurate if*** $\varepsilon_{accu} > \varepsilon$ ***(relative uncertainty of measurement).***

Remember that relative uncertainty $\varepsilon = \frac{\Delta C}{C_{Av}} x100 \; \%$ ***is different from*** ε_{accu}.

Note: For an *a big number of measured data* and for *accurate measurement*, the *average* should fit to the *expected value* of parameter and ε_{accu} ***should be practically zero. Meanwhile, for*** *a big number of measured data* ***ε tents to a fixed value different from zero. Actually, ε can never be equal to zero.***

CALCULATION OF UNCERTAINTY PROPAGATION BY USE OF DIFFERENTIALS

-The derivative of a function ***y = y(x)*** is noted ***y'(x) = dy/dx*** (9)
In this expression, the ***differentials dx*** and ***dy*** represent the infinitesimal small change of quantities ***x, y***.
Based on expression (1) we may relate these differentials ***dy = y'(x)* dx*** (10)
The ***mathematical differentials*** are extremely small and ***non measurable*** but if we assume that the derivative of function ***y'*** *remains almost constant* ***in a small but measurable region Δx of x- values, we can write the relation (10) in the form*** ***Δy = y'(x)*Δx*** (11)
This relation is used very efficiently in physics for error propagation calculation (type 3).
If the function "y" has two variables x_1, x_2, then expr. (3) becomes $\Delta y = y'_{x1}*\Delta x_1 + y'_{x2}*\Delta x_2$ (12)

-**Ex_1**. Measuring the length of an object by using a meter stick with the ***smallest unit is 1cm***. The procedure consist in reading the positions x_1, x_2 of two object ends and calculating its length as $L = x_2 - x_1$
In this case we get *0.5cm* absolute *uncertainty* during the *reading process*. So, if we read x_1= **56cm** we have $\Delta x_1 = 0.5cm$ ***and if we*** **read** x_2**= 96cm** ***we have*** $\Delta x_2 = 0.5cm$***, too.*** Then, we calculate the ***best estimation*** for the length (*y- function*) of object as ***L = 96 - 56 = 40cm.*** To find ***ΔL*** we use eq.(12). **As** the function is $L = x_2 - x_1$ it comes out that $L'_{x1} = -1$, $L'_{x2} = 1$ and $\Delta L = +1*\Delta x_2 - 1*\Delta x_1 = \Delta x_2 - \Delta x_1$
Before proceeding with numerical calculation we substitute ***"-" by "+" because we wants to refer to the worst case of uncertainty.*** *This is known as the* ***conservative approach*** *in* ***uncertainty calculations.***
So, we get $\Delta L = \Delta x_2 + \Delta x_1 = 0.5 + 0.5cm = 1cm$ ***and*** $\varepsilon = \Delta L/L*100\ \% = 1/40*100 = 2.4\ \%$

-**Ex_2**. We measure the period of an oscillation and get ***T = 5.5 ± 0.5s***.
Meanwhile, for the purposes of the study, we need to calculate an estimation for the quantity $y = T^2$.
In this case, we may proceed quickly by using the derivative ***y'= dy/dT = 2T*** and ***dy =2TdT***. Then, in physics, $\Delta y = 2T_{best}\Delta T$. Finally $y = y_{best} \pm \Delta y = 5.5^2 \pm 2*5.5*0.5 = (30.25 \pm 5.5)$ [s^2]

Remember that the use of max-min method requires a special attention to the form of mathematical expression. If the considered expression contains many variables, often one prefers to use the differential method. The two following examples make easier to understand some advantages of differential method.

Ex_3. The measurements results for three physical parameters are **A= 5.1 ± 0.3; B = 25 ± 1; C = 3.45 ± 0.05.** Calculate the ***average, maximum, minimum*** values and ***relative uncertainty*** for $y = A*B - \frac{B*C^2}{A}$. **Pay attention to sig. figures and rounding off rules.**

3-a) **Differential method** *(one starts by mathematical differential)*

- Calculate the mathematical differential of function y;

$$dy = dA*B + AdB - dB\frac{C^2}{A} - \frac{B*2C*dC}{A} - B*C^2(-1)A^{-2}dA$$

- Convert *mathematical differentials* to *physical differentials* i.e. ***physical uncertainties*** and apply the ***conservative principle*** *(* everywhere + sign)

$$\Delta y = \Delta A*B + A*\Delta B + \frac{C^2}{A}*\Delta B + \frac{2BC}{A}*\Delta C + \frac{BC^2}{A^2}*\Delta A$$

- Substitute the known *best* and *absolute uncertainty* values for A , B and C into this expression

$$\Delta y = 0.3 * 25 + 5.1 * 1 + \frac{3.45^2}{5.1} * 1 + \frac{2*25*3.45}{5.1} * 0.05 + \frac{25*3.45^2}{5.1^2} * 0.3$$

$$\boldsymbol{\Delta y} = \frac{7.5}{1SF = 0Dec} + \frac{5.1}{1SF = 0Dec} + \frac{2.3338}{1SF = 0DEC} + \frac{1.691}{1SF = 0DecC} + \frac{3.432}{1SF = 0DecC} = \frac{20.0568}{0Dec = 1SF} \equiv \mathbf{20}$$

Also, $\boldsymbol{y_{Best}} = 5.1 * 25 - \frac{25*3.45^2}{5.1} = \frac{127.5}{2SF = 0Dec} - \frac{58.345}{2SF = 0Dec} = \frac{69.15}{0Dec} = \mathbf{69}$

$\boldsymbol{y = y_{Best} \pm \Delta y = 69 \mp 20}$; $\boldsymbol{y_{Min}} = y_{Av} - \Delta y = 69 - 20 = \mathbf{49}$; $\boldsymbol{y_{Max}} = y_{Av} + \Delta y = 69 + 20 = \mathbf{89}$

And, the relative uncertainty is $\boldsymbol{\epsilon} = \frac{\Delta y}{y_{Av}} * 100\% = \frac{20}{69} * 100 = \mathbf{29}\% \equiv \mathbf{30}\%$

3-b) **<u>Max-Min method</u>** (One must pay special attention to the form of math. expression)

$$\boldsymbol{y_{Best}} = 5.1 * 25 - \frac{25 * 3.45^2}{5.1} = \frac{127.5}{2SF = 0Dec} - \frac{58.345}{2SF = 0Dec} = \frac{69.15}{0Dec} = \mathbf{69}$$

$$\boldsymbol{y_{Min}} = A_{Min} * B_{Min} - \frac{B_{Max} * C_{Max}^2}{A_{Min}} = 4.8 * 24 - \frac{26 * 3.5^2}{4.8} = \frac{115.2}{2SF = 0Dec} - \frac{66.3541}{2SF = 0Dec} = \frac{48.845}{0Dec = 1SF} \equiv \mathbf{49}$$

$$\boldsymbol{y_{Max}} = A_{Max} * B_{Max} - \frac{B_{Min} * C_{Min}^2}{A_{Max}} = 5.4 * 26 - \frac{24 * 3.4^2}{5.4} = \frac{140.4}{2SF = 0Dec} - \frac{51.3777}{2SF = 0Dec} = \frac{89.022}{0Dec = 1SF} \equiv \mathbf{89}$$

$\boldsymbol{\Delta y} = (y_{Max} - y_{Min})/2 =$ (89-49)/2 = **20** and $\boldsymbol{\epsilon} = \frac{\Delta y}{y_{Av}} * 100\% = \frac{20}{69} * 100 = \mathbf{29}\% \equiv \mathbf{30}\%$

Ex_4. The data for three physical parameters are A= **5.1** ± **0.3; B = 25** ± **1; C = 3.45** ± **0.05.**

Calculate the ***average, maximum, minimum*** values and ***relative uncertainty*** for $\boldsymbol{y = \frac{A^2 * C^5}{B^3}}$.

<u>*In this case the expression contains **only** the **product** and **the power** of parameters.*</u>

4a) <u>Differential method starting by natural logarithms</u>

-Calculate the best value $\boldsymbol{y_{Best}} = \frac{A_{Best}^2 * C_{Best}^5}{B_{Best}^3} = \frac{5.1^2 * 3.45^5}{25^3} = \frac{26.1 * 488.7597966}{15625} = \frac{0.81642436}{2SF} \equiv \frac{\mathbf{0.82}}{2SF}$

- Take the natural logarithm of expression $lny = lnA^2 + lnC^5 - lnB^3 = 2lnA + 5lnC - 3lnB$

- Take the differential of both sides $\frac{dy}{y} = 2\frac{dA}{A} + 5\frac{dC}{C} - 3\frac{dB}{B}$

- Convert *mathematical differentials* to *physical differentials* i.e. ***physical uncertainties*** and apply the ***conservative principle*** *(*put everywhere + sign) $\frac{\Delta y}{y} = 2\frac{\Delta A}{A} + 5\frac{\Delta C}{C} + 3\frac{\Delta B}{B}$

$$\frac{\boldsymbol{\Delta y}}{\boldsymbol{y}} = 2\frac{0.3}{5.1} + 5\frac{0.05}{3.45} + 3\frac{1}{25} = \frac{0.117647}{1SF = 1Dec} + \frac{0.07246}{1SF = 5Dec} + \frac{0.12}{1SF = 1ec} = \frac{0.310107}{2Dec} \equiv \mathbf{0.3}$$

- Calculate the *absolute uncertainty* as $\boldsymbol{\Delta y} = y_{Av} * 0.3 = 0.82 * 0.31 = \frac{0.254}{2SF} \equiv \mathbf{0.25}$

So, the result is $\boldsymbol{y = 0.82 \pm 0.25}$ *and* $\boldsymbol{\varepsilon} = \frac{\mathbf{0.25}}{\mathbf{0.82}} * \mathbf{100}\% = \frac{\mathbf{0.3048}}{\mathbf{2SF}} \equiv \mathbf{30}\%$

4b) Max-Min method

$$y_{Best} = \frac{A^2_{Best}*C^5_{Best}}{B^3_{Best}} = \frac{5.1^2*3.45^5}{25^3} = \frac{26.1*488.7597966}{15625} = \frac{0.81642436}{2SF} \equiv \frac{\mathbf{0.82}}{2SF}$$

$$y_{Max} = \frac{A^2_{Max}*C^5_{Max}}{B^3_{Min}} = \frac{5.4^2*3.5^5}{24^3} = \frac{29.16*525.21875}{13824} = \frac{1.1078833}{2SF} \equiv \frac{\mathbf{1.1}}{2SF}$$

$$y_{Min} = \frac{A^2_{Min}*C^5_{Min}}{B^3_{Max}} = \frac{4.8^2*3.4^5}{26^3} = \frac{23.04*454.35424}{17576} = \frac{0.595603}{2SF} \equiv \frac{\mathbf{0.60}}{2SF}$$

$\Delta y = (y_{Max} - y_{Min})/2 =$ (1.1 - 0.6) / 2 = **0.25** and $\epsilon = \frac{\Delta y}{y_{Av}} * 100\% = \frac{0.25}{0.82} * 100 = \mathbf{30}\%$

Remember: The result of a calculation cannot have higher precision than any of terms.
So, one starts by doing the mathematical calculations and follows by keeping at the result the number of digits that fits to the less precise term (conservator principle). In practice, one has to note the *significant figure* and the *number of decimals* for each term. Then, depending on the form of mathematical expression, one applies the rules of Sig.Fig.and DEC to identify the uncertain digit at the result. Finally, one rounds off the result.