11.1 APPLICATION OF AMPERE'S LAW IN SYMMETRIC MAGNETIC FIELDS

- If one knows that a magnetic field has a symmetry**,** one may calculate the magnitude of $\vec{B}$ by use of

<u>Ampere's law</u>: ***The integral of scalar product*** $\vec{B}*\vec{dl} = Bdl\cos\theta$ ***over a closed path is***

$$\oint_{Closed\_path} \vec{B}*d\vec{l} = \mu_0 I_{net} \qquad (1)$$

I_{net} is the ***net current*** flowing ***through the area <u>inside</u> closed path***.

The right hand rule fixes the positive sense of circulation on the closed path as follows; if the thumb points along direction of net current (I_{net}) sense, the ***curled fingers*** indicate the ***positive sense of circulation*** on the closed ***path***.

Examples. ***a)*** Find the magnitude of field due to a current ***I*** flowing in a straight long wire (fig.1). Due

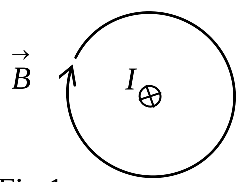


Fig.1
Cylindrical symmetry of $\vec{B}$

to the cylindrical symmetry, $\vec{B}$ has the *same magnitude* at all points at distance ***R*** from the wire and it is *tangent* to the circle passing by these points; the right hand rule gives the same direction for the sense "+" of circulation and $\vec{B}$ (fig.1).One selects a circular path and applies express. (1)

$$\oint_{circle\_R} \vec{B}*d\vec{l} = \int_{circle\_R} Bdl = B*2\pi R = \mu_0 I \rightarrow\rightarrow B = \frac{\mu_0 I}{2\pi R} \qquad (2)$$

Note that the vector $\vec{dl}$ has the same direction as vector $\vec{B}$.

b) Find the magnetic field due to an ***ideal infinite solenoid*** that has ***n <u>turns/m</u>*** and carries the current ***I***. In ideal infinite long solenoids the field ***outside the solenoid*** is ***zero*** (Fig. 2) and ***inside*** it is pretty much ***uniform*** (same $\vec{B}$ everywhere). Its direction (to the right as shown) is found by the right hand rule.

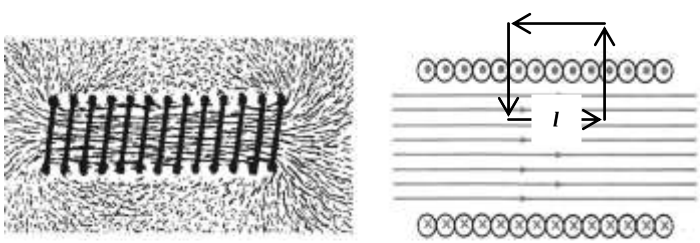


Fig.2

One may *select* a closed *square* (with *side l*) *path* like that shown in fig.2 ("+" sense is *fixed by right hand rule*) and apply expr. (1). As $\vec{B}*\vec{dl} = Bdl$ along field lines inside ; $\vec{B}*\vec{dl} = 0$ outside or along the "vertical" sides it comes out that $\oint \vec{B}*\vec{dl} = B*l$

As, there is ***nl*** wires in length ***l***; so the *net current* through the square path is ***nlI.*** Then, from (1) we get

$$B*l = \mu_0 I_{net} \rightarrow\rightarrow B*l = \mu_0 n*l*I \qquad \text{and} \qquad B = \mu_0 nI \qquad (3)$$

c) Find the magnetic field of a *toroidal coil* with ***N*** turns each carrying the same current ***I*.**

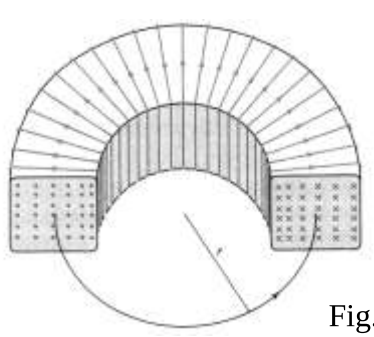


Fig.3

A quick observation shows that the magnetic field lines are circles passing perpendicular to the *toroid* sections. We select a circular path with radius ***r*** as shown in fig .3 and apply the Ampere's law. As the field *magnitude* "B" is the same around the circuit, we get:

- inside wired section $\quad B*2\pi r = \mu_0 NI \rightarrow\rightarrow B = \frac{\mu_0 N}{2\pi r} I \qquad (4)$
- outside toroid $\quad$ B = 0 because the net current is ***–N*I + N*I = 0***
- inside toroid $\quad$ B = 0 because net current is zero. (***I = 0***)

11.2 INDUCTANCE

- A carrying current solenoid () is the best tool to ***build, change and control*** a magnetic field. The main property of a solenoid is the *reaction versus the change of magnetic flux passing through it.*

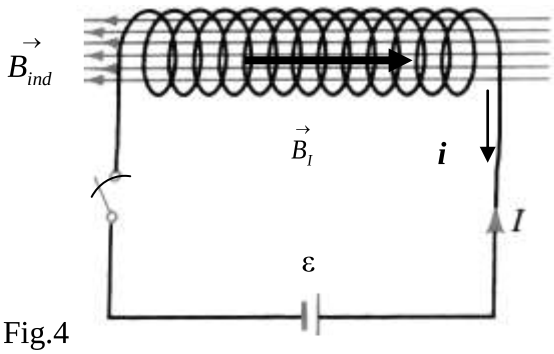


Fig.4

Fig.4 presents what happens at a solenoid *a few moments* after the switch is turned on. The current "***I***", due to source "***ε***", builds inside solenoid the field $\vec{B}_I$ "*directed right side* ". This means an increase of magnetic flux (*it was 0*) inside the solenoid. The Lenz law tells that the " *solenoid will react* " by ***inducting*** a current " ***i*** " such that the related magnetic field $\vec{B}_{ind}$ be opposite to $\vec{B}_I$. So, the inducted current "***i***" flows in the opposite sense with respect to "***I***".

The reaction of solenoid does not *<u>allow instantaneous set of current</u>* value in circuit as given by Ohm's law ($I = \varepsilon / R_{sol}$). This kind of reaction happens when the switch is turned off, too, but the direction of inducted current "***i***" is the same as that of ***I*** and $\vec{B}_{ind}$ has the same direction as $\vec{B}_I$; *<u>the circuit reaction does not " want " to leave the flux decrease</u>*. In the first case, one observes a *gradual* increase (Fig. 5a) of the net current and in the second case one observes a *gradual* decrease (Fig5.b) of current in circuit.

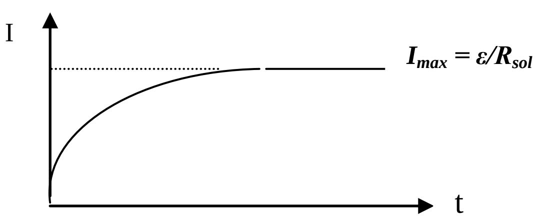
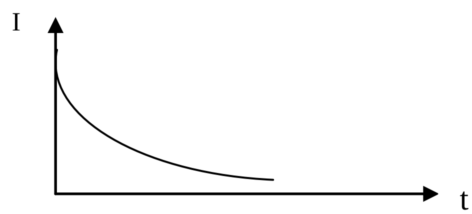


Fig5.a Switch turn on



5.b Switch turn off

- The phenomena presented in the upper paragraph is known as ***self-induction because it*** *is related to the flux built by the solenoid and passing through itself*. This is a phenomenon that happens in all circuits; the presence of a solenoid into the circuit just makes it more pronounced.
If there is a magnetic flux $\boldsymbol{\Phi_L}$ through *a single turn* (*loop) of a coil (solenoid),* then, the net flux through its ***N*** *turns is* ***N* Φ***$_L$. The ***flux linkage*** *through the coil* is $\boldsymbol{\Phi_c = N^* \Phi_L}$ [*Weber*] (5)

-In many situations (ex. transformers), two coils are arranged in such way that the magnetic flux from coil " 2 " through coil " 1 " (*and vice-versa* Fig.6) be maximal. Note that each coil is part of a different circuit and carries a different current (I_1 , I_2). In this situation, the flux through ***<u>each turn</u>*** of ***coil_1*** (*let's refer to coil "1"*) is constituted by two components: $\boldsymbol{\Phi_L}$ due to current I_1 and $\boldsymbol{\Phi_M}$ due to current I_2 through second coil. Then,

$$\Phi = \Phi_L + \Phi_M \tag{6}$$

the net ***flux linkage*** through ***turns coil_1*** (with N_1 turns) is

$$\Phi_{c-1} = N_1\Phi = N_1\Phi_L + N_1\Phi_M \tag{7}$$

Therefore, the expression for the ***efm*** *inducted* in ***coil_1*** is

$$\varepsilon_1 = -\frac{d}{dt}\Phi_{c-1} = -\frac{d}{dt}(N_1\Phi_L) - \frac{d}{dt}(N_1\Phi_M) = \varepsilon_{L-1} + \varepsilon_{M-1} \tag{8}$$

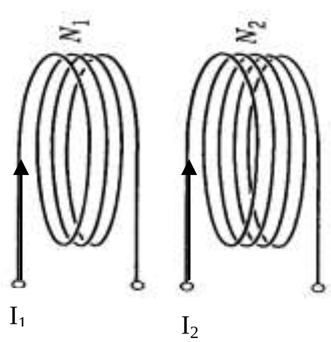


Fig.6

- So far, one uses several parameters (*loop diameter, coil length, number of turns and current*) to express the E.M. induction behaviour of coils. Next, one introduces a single parameter to measure the effect of coils geometry on its E.M. induction. Let's consider first the ***self-induction*** of coil "1" presented by the factor "$N_1 * \Phi_L$" at expression (7). The field "B_1" and *its flux* through *one turn* "Φ_L" are <u>proportional</u> to the magnitude of current "I_1". As the other contributions to "$N_1 * \Phi_L$" depend on *number of loops* and the *geometry* of coil_1, one puts them together into a single parameter "L_1" known as ***coil self-inductance.***

Then, the ***self-inducted flux linkage*** in coil " 1 " can be expressed as $$N_1\Phi_L = L_1 * I_1 \qquad (9)$$

L_1 [**H**-*Henry*] is a parameter that depends on coil "1" geometry (*length, diameter, number of turns*) and defines the ***"self-inducted emf "*** by relation

$$\varepsilon_{L-1} = -\frac{d}{dt}(N_1\Phi_L) = -\frac{d}{dt}(L_1 * I_1) = -L_1\frac{dI_1}{dt} \qquad (10)$$

The direction of ε_L is such that the related current ***<u>opposes</u>*** ***the*** ***<u>changes of current I</u>*** in circuit.

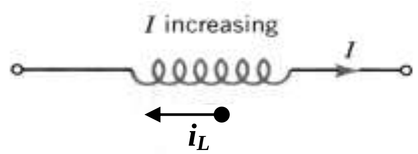


Fig. 7.a

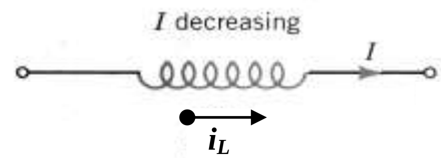


Fig. 7.b

- The " ***mutual inductance M*** " measures the *geometrical effects* on the part of flux through first coil due to current in second coil. So, $$N_1\Phi_M = M * I_2 \qquad (11)$$

The parameter " ***M*** " depends only on the *geometry of the whole set of two coils*. It has the same value when calculating the effect of *current* in *coil **1*** on flux through *coil **2**, i.e.* $N_2\Phi_M = M * I_1$. One may figure out easily that "***M***" is *larger* when the *coils are closer* to each other and when they have the *same central axe*. Based on (11) one gets that ***emf in coil 1*** due to change of *flux generated from coil 2* is

$$\varepsilon_{M-1} = -\frac{d}{dt}(N_1\Phi_M) = -\frac{d}{dt}(M * I_2) = -M\frac{dI_2}{dt} \qquad (12)$$

Then, expression (8) can be written as

$$\varepsilon_1 = \varepsilon_{L-1} + \varepsilon_{M-1} = -L_1\frac{dI_1}{dt} - M\frac{dI_2}{dt} \qquad (13)$$

11.3 LR CIRCUITS

-A real ***inductor L*** has always a resistance; in the following model one refers to an *ideal inductor* (*zero resistance)* and includes the inductor resistance into external resistor ***R***. To find the evolution of current in a RL circuit, one may apply the Kirchhoff rule to the circuit in Fig. 8 at a moment "t" when there is a self-*inducted* ***emf*** "ε_L" with magnitude " ***LdI/dt*** " acting in circuit (this is *not a steady state situation*).

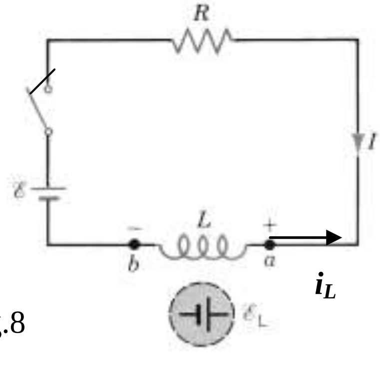


Fig.8

a) Once the switch is turned on, the current " ***I*** " starts flowing along the direction shown and builds up a magnetic field in the coil. This <u>*increases the magnetic flux through coil*</u> induces an ***emf*** . This ***emf*** produces current "i_L" directed in opposite sense to ***I.*** Consequently, there is only a moderate increase of current "***I***" in circuit. Kirchhoff's rule gives:

$$\varepsilon - IR - \varepsilon_L = 0 \rightarrow \varepsilon - IR - L\frac{dI}{dt} = 0 \Rightarrow\Rightarrow\Rightarrow \frac{\varepsilon}{R} - I = \frac{L}{R}\frac{dI}{dt} \qquad (14)$$

To solve equation (14) one introduces a new variable $y = \frac{\varepsilon}{R} - I$ (15)

Note: The induction effect is counted by ε_L (no need to include "i_L" in calculation)

Time derivative of (15) gives $\frac{dy}{dt} = -\frac{dI}{dt}$ (16) By substituting (14) and (15) into equation (13) one get

$$y = \frac{L}{R}(-\frac{dy}{dt}) \to \frac{dy}{y} = -\frac{R}{L}dt \to \int_{y0}^{y}\frac{dy}{y} = -\frac{R}{L}\int_{0}^{t}dt \Rightarrow \ln y - \ln y_0 = -\frac{R}{L}t \to \ln\frac{y}{y_0} = -\frac{R}{L}t \Rightarrow and \Rightarrow$$

$$y = y_0 e^{-\frac{R}{L}*t} \qquad (17)$$

At t = 0, I=0 and $y_0 = \mathcal{E}/R$. Then, by noting $\mathcal{E}/R \equiv I_{max}$ the expression (16) gives

$$y = \frac{\varepsilon}{R} - I = \frac{\varepsilon}{R}e^{-\frac{R}{L}*t} \to I_{max} - I = I_{max}e^{-\frac{R}{L}*t} \to\to I = I_{max}(1 - e^{\frac{-t}{L/R}}) \qquad (18)$$

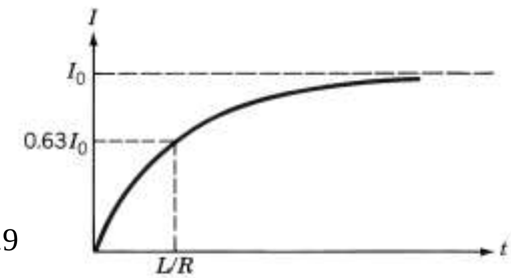


Fig.9

After defining the *time constant* as $\tau = L/R$ (19)

the expression (17) transforms to $I = I_{max}(1 - e^{-\frac{t}{\tau}})$ (20)
At time t = τ the current is $0.63*I_0$.

A similar function describes the increase of charge Q in a capacitor in a RC circuit but in that case τ = RC.

- Once the current gets to value ***Io***, there is no more ***inducted efm*** in the circuit because the flux remains constant. The flux through the coil changes when the source ***ε*** is switched off. Note that, in this case, one needs a closed circuit to observe the current evolution. That's why we refer to the scheme presented in Fig.10a where the switch S_2 is turned on as the switch S_1 is turned off. In this situation, the ***inducted emf***

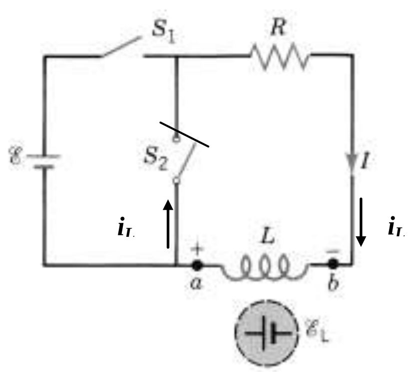


Fig.10.a

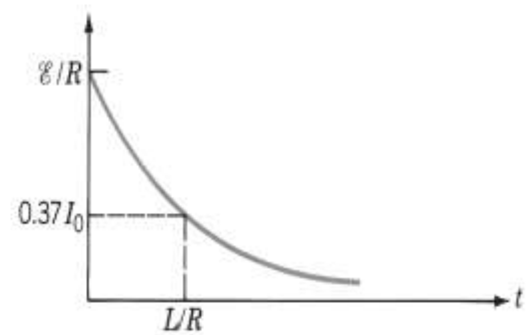


Fig.10.b

opposes the decrease of magnetic flux through coil by producing a current "$\boldsymbol{i_L}$ " along the same sense as the current "***I***" (ε_L **= - *L*dI/dt*** is positive because ***dI/dt < 0***). The Kirchhoff rule for this situation gives

$$-IR + \varepsilon_L = 0 \to -IR - L\frac{dI}{dt} = 0 \; and \; \frac{dI}{I} = -\frac{R}{L}dt \to \int_{I_{max}}^{I}\frac{dI}{I} = \int_0^t\left(-\frac{R}{L}\right)dt$$

and

$$ln\frac{I}{I_{max}} = -\frac{t}{L/R} \to\to I = I_{max}e^{-\frac{t}{L/R}}$$ because at t = 0 $I = I_{max} = \frac{\varepsilon}{R}$

Then, with notation τ = L/ R one get $$I = I_{max} * e^{-\frac{t}{\tau}} \qquad (21)$$

The graph in figure 10.b presents the decrease of current following expression (20).
For ***t = τ = L/R***, the current falls down by 63% of its initial value I_0 and gets to 0.37% of I_0.

11.4 ENERGY STORED INSIDE AN INDUCTOR

-Let's consider another time the circuit in fig.8 a few moments after the switch is turned on. At a moment " t ", the relation (14) is written as $\varepsilon - I(t)R - L\dfrac{dI(t)}{dt} = 0 \Rightarrow\Rightarrow \varepsilon = I(t)R + L\dfrac{dI(t)}{dt}$ (22)

Note that ***I(t)*** is smaller than the current value at steady state ***($I_0 = \varepsilon / R$)***.

By multiplying by *I(t)* both sides of (22) one get $I(t)\varepsilon = I^2(t)R + LI(t)\dfrac{dI(t)}{dt}$ (23)

By referring to an *ideal emf* source, the product "***I(t)*ε***" would present the ***power supplied*** by source into the circuit <u>*at the moment **t***</u>. The factor ***$I^2(t)$*R*** is the ***power dissipated thermally*** into the resistor <u>*at this moment*</u>**.** Then, the term $LI(t)\dfrac{dI(t)}{dt}$ would present *the **power** being **supplied to the inductor*** <u>*at moment t*</u> . By referring to the direction of current ***I*** and the polarity of inducted emf $\boldsymbol{\varepsilon_L}$ at Fig.8, one might figure out that the source "***ε***" is supplying energy in circuit and the ***"inducted emf source ε_L"*** is *storing energy inside the inductor "L" as a **magnetic field energy $U_L(t)$***.

So, *the* <u>*increase rate*</u> of magnetic energy stored in inductor *(i.e. dU_L/dt) is equal to the power delivered by the **source** "ε" to the "ε_L"*. One get the energy stored at inductor from t=0 till time " ***t*** " as follows

$$\frac{dU_L(t)}{dt} = LI(t)\frac{dI(t)}{dt} \rightarrow\rightarrow dU_L = LI(t)dI(t) \Rightarrow \int_0^t dU_L = L\int_0^t I(t)dI \Rightarrow U_L = L\frac{I^2(t)}{2}$$

So, the energy stored in inductor at moment "t" during transitory period is $U(t)_L = \frac{1}{2}LI(t)^2$ (24)

and at any moment after transitory period (when I= $\boldsymbol{I_0 = I_{max} = \varepsilon / R}$) $U_L = \frac{1}{2}LI_{max}{}^2$ (25)

This <u>*energy* is *stored inside the inductor*</u>. It is due to the presence of <u>*magnetic field inside the inductor and it remains the same during all the steady state in circuit*</u>.

Remember that the electric field energy stored inside a capacitor at the end of the transitory period is $U_c = \frac{1}{2}\frac{Q^2}{C}$ or $U_c = \frac{1}{2}CV^2$ and one uses the letter ***U*** to indicate that this is a type of "*potential energy*".

- Let's see the case of a **solenoid** with ***n*** **[**turns/m], length ***l*** [m], cross section area ***A*** and current *I* [A].

The magnitude of field inside solenoid (*ideal model, see relation 3*) is $B = \mu_0 nI$ (26)

The magnetic flux through it is $\Phi = N(A*B) = (n*l)*(A*B) = n*l*A*\mu_0 nI = \mu_0 * n^2 * A*l*I = L*I$ (27)

So, the *self inductance* for a solenoid is $L = \mu_0 * n^2 * A*l$ (28)

By substituting (28) at relation (24) and isolating I from (26) as $I = \frac{B}{\mu_0 n}$, one gets

$$U_L = \frac{1}{2} L I^2 = \frac{1}{2} \mu_0 n^2 A * l \frac{B^2}{\mu_0^2 n^2} = \frac{B^2}{2\mu_0} A * l \qquad (29)$$

This energy is stored inside the whole volume (*A*l*) of solenoid. So, the energy stored inside the unit

volume i.e. the ***density of magnetic energy*** is

$$u_B = \frac{1}{2\mu_0} * B^2 \qquad (30)$$

-It is interesting to mention that the amount of magnetic energy given by expression (29) get converted into electric energy of the electric spark produced at the switch while it is turned off.

Example: The *ignition coil in an automobile makes use of this effect to fire the spark plug.*

Important Notes: - The expression (30) is valid for ***any magnetic field***.

- It is very similar to the density of electric energy (***any electric field*** $u_E = \frac{\varepsilon_0}{2} * E^2$)
- As expected (from *wave nature of fields*) these energy expressions are ~ to the *square of field strength*.