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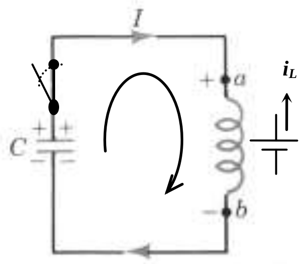
12.1 OSCILLATION OF ELECTRIC VARIABLES IN A LC CIRCUIT

- Let's consider a circuit that contains only a capacitor C and an ideal inductor L in series (see fig.1). Assume that at t = 0 the capacitor has a charge "Q_0". At this moment one turns the switch "on" and the charge of capacitor C starts flowing in circuit by building a current "***I*** " directed as shown in the figure. The increase of flux into the inductor gives rise to an inducted $\boldsymbol{\varepsilon_L}$ (*and related* i_L) with the "opposite" polarity as shown in figure. The second rule of Kirchhoff (start from " – " plate of capacitor) gives

$$V_c(t) - \varepsilon_L = \frac{Q(t)}{C} - L\frac{dI(t)}{dt} = 0 \qquad (1)$$



As $\frac{dQ}{dt} < 0$ and I > 0 , the current I is equal to "-" *change rate* of charge Q in capacitor plates. So, one gets $I(t) = -\frac{dQ(t)}{dt}$ (*)

Then, by substituting relation (*) in eq.1 one gets

Fig.1

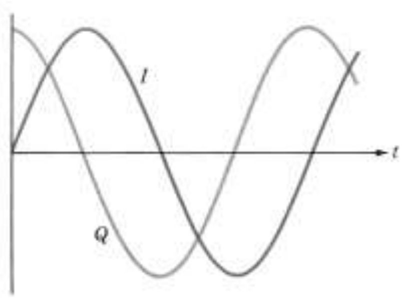
$$\frac{Q(t)}{C} + L\frac{d^2}{dt^2}Q(t) = 0 \to\to \frac{d^2}{dt^2}Q(t) + \frac{1}{LC}Q(t) = 0 \qquad (2)$$

By noting $\omega_0^2 = \frac{1}{LC}$ _i.e._ $\omega_0 = \sqrt{\frac{1}{LC}}$, the eq.(2) takes the form $\frac{d^2}{dt^2}Q(t) + \omega_0^2 Q(t) = 0$ (3)

This is the equation of SHO and its solution has the form $Q(t) = Q_0\sin(\omega_0 t + \varphi)$ (4)

At t = 0, Q = Q_0 that fixes φ = π/2. So, $Q(t) = Q_0\sin(\omega_0 t + \pi/2) = Q_0\cos(\omega_0 t)$ (5)

Then $I(t) = -\frac{dQ(t)}{dt} = Q_0\omega_0 \sin\omega_0 t = I_0 \sin\omega_0 t$ i.e. with $\boldsymbol{I_0 = Q_0 * \omega_0}$ $I(t) = I_0\sin\omega_0 t$ (6)



So, it comes out that: ***if there is some energy stored in a LC circuit, the circuit will generate a SHO with <u>circular frequency</u>*** $\omega_0 = \sqrt{\frac{1}{LC}}$ for the ***current,*** the ***voltage*** and the ***charge in capacitor***.

Fig. 2 shows the variations of ***Q(t)*** and ***I(t)*** in time; the charge in capacitor and its voltage $\boldsymbol{V_C = Q/C}$ are ***<u>advanced</u>*** by ***π/2*** versus the current in circuit. When $\mathbf{Q}_C$ and $\mathbf{V}_C$ are maximum the ***current in circuit is zero*** (and vice versa).

At *<u>t = 0, all the energy</u>* is electric and stocked *<u>inside the capacitor</u>* :

$$U_{E-max} = \frac{Q_0^2}{2C} \qquad (7)$$

Fig.2

For t > 0, a part of the energy is stocked as *electric energy inside the capacitor* $U_E = \frac{Q_0^{\,2}}{2C}\cos^2\omega_0 t$ and the other part as *magnetic energy inside the inductor* $U_B = \frac{1}{2}LI_0^{\,2}\sin^2\omega_0 t$; so the <u>total energy in circuit</u> is $U = \frac{Q_0^2}{2C}\cos^2(\omega_0 t) + \frac{LI_0^{\,2}}{2}\sin^2(\omega_0 t)$ (8)

Figure 3 presents the evolution of each part of of energy with time. Note that their sum is ***all time constant*** (like at any SHO) and when one of them is maximum the other one is zero.

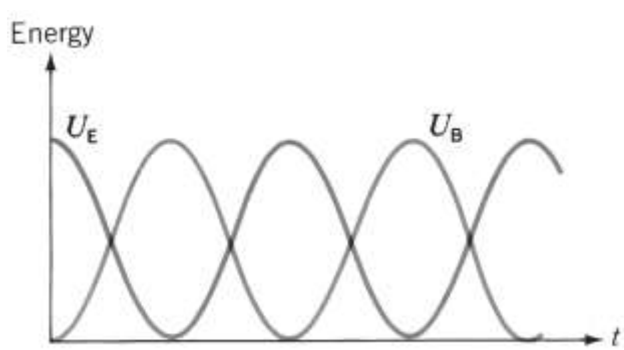


Fig 3

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12.2 DAMPED OSCILLATIONS IN A CIRCUIT C, L, R

- In previous section we considered an *ideal inductor* with zero resistance and we got a result which is *ideal*; i.e. *SHO for Q, I and U in the circuit*. This means oscillations that follow to infinite time. In the case of a real inductor, there is a resistance (*at least that of inductor*) in circuit and one deals with a circuit with C, L, R in series (fig.4). Assume that, at t = 0s, the capacitor has its maximum charge Q_0.

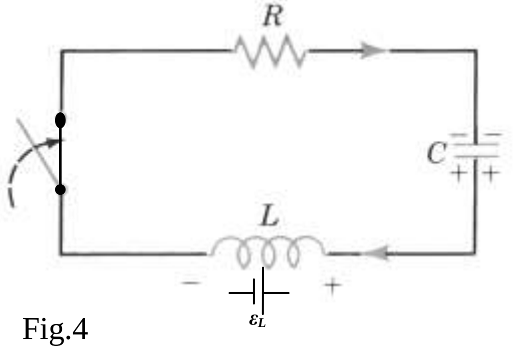

Fig.4

At this moment (t = 0) one turns the circuit " on ". By applying the second rule of Kirchhoff for a moment t > 0 , one get

$$\frac{Q(t)}{C} - L\frac{dI}{dt} - IR = 0 \qquad (9)$$

By using the expression (*) for the current, one get

$$L\frac{d^2Q(t)}{dt^2} + R\frac{dQ(t)}{dt} + \frac{Q(t)}{C} = 0 \qquad (10)$$

This is the equation of a damped harmonic oscillation. One might remember the equation for damped harmonic motions : $m\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + kx(t) = 0$ where ***b*** is the damping coefficient.

By referring to DHO (see oscillations in NYC folder) , one finds out that the solution of (10) has the form

$$Q(t) = Q_0 * e^{-\left(\frac{R}{2L}\right)*t}\sin(\omega' t + \varphi) \qquad (11)$$

where the ***damped circular frequency*** is $\omega' = \sqrt{\omega_0^2 - \left(\frac{R}{2L}\right)^2}$ (12)

The <u>un-damped circular frequency</u> for a C, L, R circuit corresponds to ***R = 0*** and is $\omega_0 = \frac{1}{\sqrt{LC}}$ (13)

- The evolution of electric parameters (***Q, I, U***) in circuit depends on the relative values of ***R, L and C***.

a) There is a <u>under damped</u> oscillation if $R < 2\omega_0 L$. ***The <u>amplitude</u> of*** oscillations for ***(Q, I, U) <u>decays exponentially</u>. Example (Fig.5a):*** $Q(t) = Q_0 * e^{-t/\tau}$ where $\tau = 2L/R$ (14)

b) There are <u>critically damped</u> oscillations if $R = 2\omega_0 L$. ***Actually, in this case there are no oscillations in circuit. All electric variables just fall to zero the fastest way (Fig.5b-downside).***

c) There are <u>over damped</u> oscillations if $R > 2\omega_0 L$. ***Even in this case there are no oscillations in circuit. All electric variables just fall to zero with time, but slowly (Fig.5b-graph upside).***

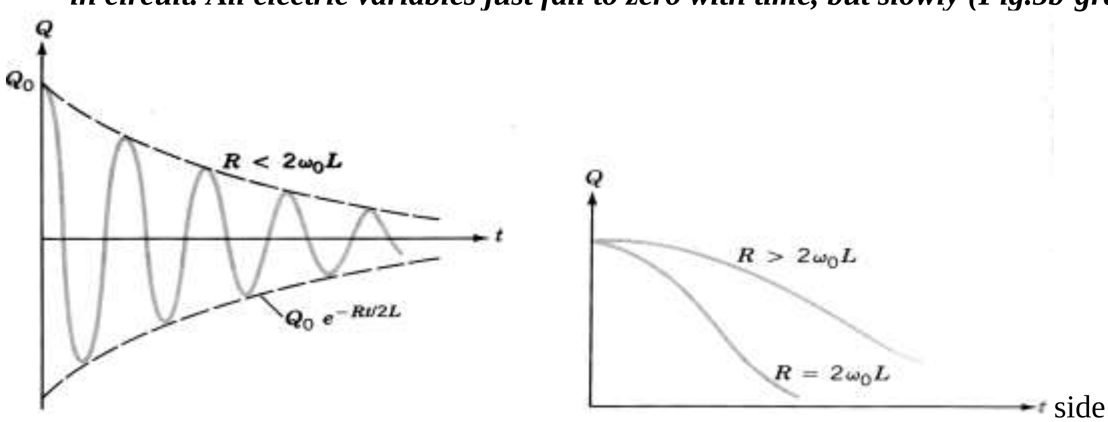

Fig.5 (a) (b)