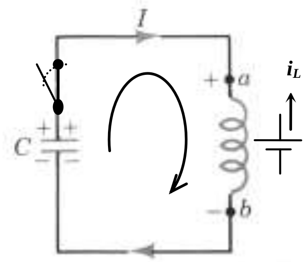
12.1 OSCILLATION OF ELECTRIC VARIABLES IN A LC CIRCUIT

- Let's consider a circuit with a capacitor C and an inductor L in series (see fig.1); the capacitor has a charge "Q_0" at t = 0. At this moment one turns the switch on and the charge of capacitor C starts flowing into the circuit; it will build a current "***I*** " as shown in fig. The increase of flux into the inductor gives rise to an inducted $\boldsymbol{\varepsilon_L}$ (*and related* i_L) with the polarity shown in figure. The second rule of Kirchhoff (start from " – " plate of capacitor) gives



$$V_c(t) - \varepsilon_L = \frac{Q(t)}{C} - L\frac{dI(t)}{dt} = 0 \qquad (1)$$

The current I is equal to "-" *change rate* of charge Q in capacitor plates.

So, as $\frac{dQ}{dt} < 0$ and $I > 0$ one gets $I(t) = -\frac{dQ(t)}{dt}$ (*)

Then, by substituting relation (*) in eq.1 one gets

$$\frac{Q(t)}{C} + L\frac{d^2}{dt^2}Q(t) = 0 \rightarrow\rightarrow \frac{d^2}{dt^2}Q(t) + \frac{1}{LC}Q(t) = 0 \qquad (2)$$

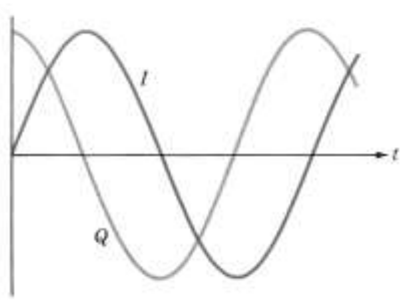
Fig.1

By noting $\omega_0^2 = \frac{1}{LC}$ _i.e._ $\omega_0 = \sqrt{\frac{1}{LC}}$, the eq.(2) takes the form $\frac{d^2}{dt^2}Q(t) + \omega_0^2 Q(t) = 0$ (3)

This is the equation of SHO and its solution has the form $Q(t) = Q_0 \sin(\omega_0 t + \varphi)$ (4)

At t = 0, Q = Q_0 that fixes φ = π/2. So, $Q(t) = Q_0 \sin(\omega_0 t + \pi/2) = Q_0 \cos(\omega_0 t)$ (5)

Then $I(t) = -\frac{dQ(t)}{dt} = Q_0\omega_0 \sin\omega_0 t = I_0 \sin\omega_0 t$. So with $\boldsymbol{I_0 = Q_0\omega_0}$ $I(t) = I_0 \sin\omega_0 t$ (6)



From expressions (1-6) comes out that: ***"if there is some energy stored in a LC circuit, the circuit will generate a SHO with <u>circular frequency</u>*** $\omega_0 = \sqrt{\frac{1}{LC}}$ for the ***current,*** the ***voltage*** and the ***charge in capacitor***.

Fig. 2 shows the variation of ***Q(t)*** and ***I(t)*** in time; the charge in capacitor and its voltage $\boldsymbol{V_C = Q/C}$ are ***<u>advanced</u>*** by ***π/2*** versus the current in circuit. When $\mathbf{Q}_C$ and $\mathbf{V}_C$ are maximum the ***current in circuit is zero*** (and vice versa). At *<u>t = 0, all the energy</u>* is electric and stocked *<u>inside the capacitor</u>* :

Fig.2

$$U_{E-max} = \frac{Q_0^2}{2C} \qquad (7)$$

For t > 0, a part of the energy is stocked as *electric energy inside the capacitor* $U_E = \frac{Q_0^2}{2C}\cos^2\omega_0 t$ and the other part as *magnetic energy inside the inductor* $U_B = \frac{1}{2}LI_0^2 \sin^2\omega_0 t$; so the <u>*total energy in circuit*</u> is $U = \frac{Q_0^2}{2C}\cos^2(\omega_0 t) + \frac{LI_0^2}{2}\sin^2(\omega_0 t)$ (8)

Figure 3 presents the evolution of each part of of energy with time. Note that their sum is ***all time constant*** (like at any SHO) and when one of them is maximum the other one is zero.

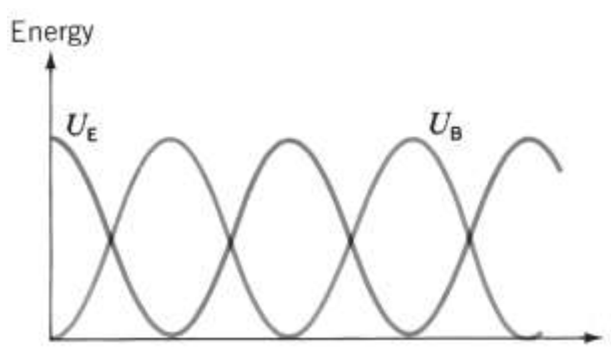


Fig 3

12.2 DAMPED OSCILLATIONS IN A CIRCUIT C, L, R

- In previous section we considered an *ideal inductor* with zero resistance and we got a result which is ideal; i.e. *SHO for Q, I and U in the circuit*. This means oscillations that follow to infinity time. Now, let's see the case of a real inductor; i.e. an inductor resistance R. In this case one has a circuit with C, L, R in series (fig.4). Assume that, initially, at t = 0s, the capacitor has its maximum charge Q_0 .

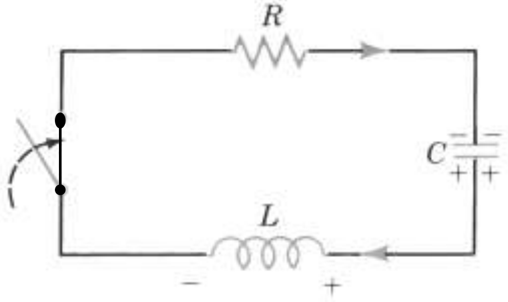


Fig.4

At this moment (t = 0) one turns the circuit " on ". Let´s apply the second rule of Kirchhoff for a moment t > 0. We will get

$$\frac{Q(t)}{C} - L\frac{dI}{dt} - IR = 0 \qquad (9)$$

By using the expression (*) for the current, we get

$$L\frac{d^2Q(t)}{dt^2} + R\frac{dQ(t)}{dt} + \frac{Q(t)}{C} = 0 \qquad (10)$$

This is the equation of a damped harmonic oscillation. One might remember the equation for damped harmonic motions : $m\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + kx(t) = 0$ where ***b*** is the damping coefficient.

By referring to DHO (NYC folder) , one finds out that the solution of (10) has the form

$$Q(t) = Q_0 * e^{-(R/2L)*t}\cos(\omega' t + \delta) \qquad (11)$$

where the ***damped circular frequency*** is $\omega' = \sqrt{\omega_0^2 - \left(\frac{R}{2L}\right)^2}$ (12)

The <u>*un-damped circular frequency*</u> for a C, L, R circuit corresponds to ***R = 0*** and is $\omega_0 = \frac{1}{\sqrt{LC}}$ (13)

- The evolution of electric parameters (***Q, I, U***) in circuit depends on the relative values of ***R, L and C***.

a) There are <u>*under damped*</u> oscillations if ***R < 2ω₀L. The <u>amplitude</u> of*** oscillations for ***(Q, I, U) <u>decays exponentially</u>. Example (Fig.5a):*** $Q(t) = Q_0 * e^{-t/\tau}$ where $\boldsymbol{\tau = 2L/R}$ (14)
b) There are <u>*critically damped*</u> oscillations if ***R = 2ω₀L. Actually, in this case there are no oscillations in circuit. All electric variables just fall to zero the fastest way (Fig.5b-down).***
c) There are <u>*over damped*</u> oscillations if ***R > 2ω₀L. Even in this case there are no oscillations in circuit. All electric variables just fall to zero with time, but slowly (Fig.5b-up).***

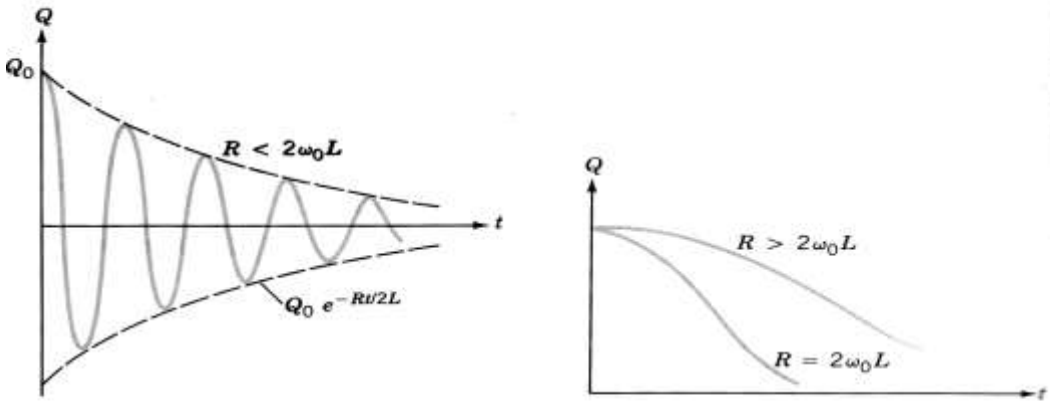


Fig.5 (a) (b)