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# 13.1 ALTERNATING CURRENT AND THE PHASOR MODEL

- If the circuit contains only DC sources, the current *flows all time in one direction*; one says that it is a **directed current (DC)**. But, in everyday activity, one deals often with ***alternating currents*** (**AC**) and AC sources(*wall socket, car alternators etc..)*. The AC is a *current that changes its direction of flowing through circuit in time*. Even the turbines of hydro/thermo centrals that provide the electric energy used in majority of human activity operate as AC sources. In North America, the wall outlets provide an AC voltage that oscillates as a harmonic function at a " *natural* " frequency ***f = 60Hz*** (i.e. $\omega = 2\pi f = 376.8\ r/s$).

-**Remember**: The AC generator is a coil that rotates at a period T ( or *frequency* $f = 1/T$ or $\omega = 2\pi f$ ) i.e. *circular frequency* ω = 2π / T inside an uniform magnetic field. The direction of the inducted ***emf*** changes with time in conformity to the sign of expression $\varepsilon = \varepsilon_0 \sin(\omega t)$. When one switches this ***emf*** into a circuit, it provides an ***harmonic AC source*** *of electricity.*
In principle**,** an AC source is not necessarily harmonic but one may show that any AC voltage *can be expressed as a sum of a* ***set of harmonic*** *AC* voltages with different frequencies (for more information see *Fourier transformations*). However, if one says AC without additional precision, this means ***harmonic AC***.

- In this chapter one uses the notations " ***i*** *and* ***υ*** " to distinguish a current and a voltage that *changes with time*. The uppercase letters ( like ***I*** and ***V*** ) will be used only for quantities that do not change with time. In general, an AC source ***provides into the circuit*** a **potential difference,** commonly called

a **voltage,** which instantaneous value is given by expression $\upsilon = \upsilon_0 \sin(\omega t + \varphi_\upsilon)$ (1)

The instantaneous current in circuit oscillates at same frequency; so $i = i_0 \sin(\omega t + \varphi_i)$ (2)

$\upsilon_o$, $i_o$ *are the maximum values(or amplitudes) ;* $\omega = 2\pi/T$ *is the circular frequency;* $\varphi_\upsilon$, $\varphi_i$ *are the phase constants.*

**Note:** The circular frequency $\omega$ is defined by the "*rotation frequency of the coil in AC generator"*.

- One refers to the ***phasor physic's model*** for the study of a parameter that oscillates in harmonic way with time, like voltage and current in a AC circuit.
*The phasor itself is a virtual vector that rotates counter clockwise at* ***circular frequency ω = 2πf*** *where* ***f = 1 / T*** *is the real frequency(in hertz) of oscillations for the physical quantity under study (see fig.1).*

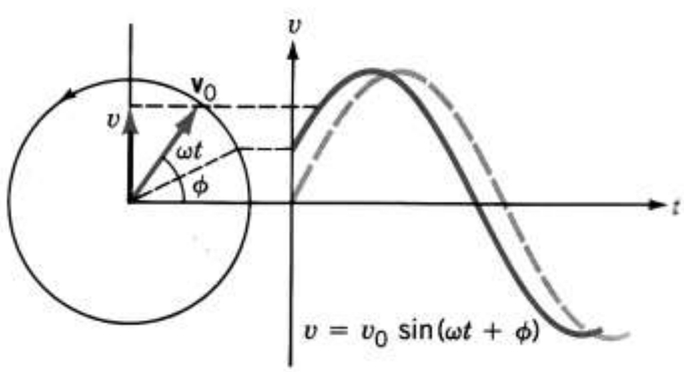



Figure 1

*The "**length**" of the phasor vector is* ***equal to the maximum value****(ex.* $\upsilon_o$ *or* $i_o$*) of oscillating parameter and the component of phasor along the vertical axis (Oy) is equal to its instantaneous value. By using the rules of trigonometric circle, one finds out easily that:*

*a) the* ***phase*** *is equal to the* ***angle Φ(t)*** $= \omega t + \varphi$
*b) if one starts(t = 0) the study from a moment when the value of considered parameter is zero, then the* ***phase constant*** $\varphi = 0$
*c) if one starts (t = 0) the study of parameter at a moment when the variable is not zero, the* ***phase constant*** $\varphi \neq 0$.

- One uses the *same* phasor diagram ***for different physical variables that oscillate at the same ω*** and this allows *to visualise the* ***phase shift*** between different AC variables (*currents and voltages)* in same circuit.

The ***phase shift*** between *voltage and current* plays a very important role in calculations for AC circuits. As mentioned, the value of the phase constant " $\varphi$ " depends on the selection of t = 0. *In a AC circuit with a resistor, an inductor and a capacitor ( R,L,C set)* ***in series***, the ***current value*** *through all elements of circuit is* ***the same*** *at any moment.*

***So, one selects t = 0 at a moment when the instantaneous current is zero and this way gets $\varphi_i = 0$.***

Then, one can write the current as

$$i = i_0 \sin(\omega t) \qquad (3)$$

and the ***phase constant*** $\varphi$ ***of instantaneous voltage*** represents the ***phase shift voltage – current***

$$\upsilon = \upsilon_0 \sin(\omega t + \varphi) \qquad (4)$$

**Notes**: a) One uses the notations $v, v_R, v_C, v_L$ for instantaneous values of voltage across the terminals of the *source, resistor, capacitor, inductor* and the sign ~ to show an AC source in circuit.
b) The *instantaneous* current and voltage *values* are counted on the same "*Oy*" axis (see fig.2).

## 13.2 RESISTOR IN AC CIRCUIT

- Let's consider the scheme in figure 2; the resistor R is connected "in series" with an ***AC source***. At a moment "***t*** ", the voltage at ***source*** terminals is ***υ(t)*** and the current through resistor is ***i(t)***. By applying the loop's rule, one get $\boldsymbol{\upsilon - \upsilon_R = 0}$. So, at any moment $\boldsymbol{\upsilon_R = \upsilon}$. As the current oscillates as $i = i_0 \sin(\omega t)$, the Ohm's law gives $v_R = iR = i_0 R sin(\omega t) = v_{0R} \sin(\omega t)$ (5)

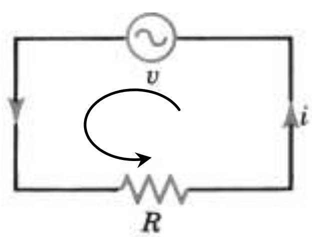



Figure 2

As $\boldsymbol{\upsilon = \upsilon_R}$, it comes out that $v = v_0 \sin(\omega t) = i_0 R sin(\omega t)$ (6)

Those relations show that for a ***resistor***:

- The *phase shift* between *applied voltage* and *current* is zero (***φ = 0***)
- The peak value of current is $i_0 = \dfrac{\upsilon_{0R}}{R} = \dfrac{\upsilon_0}{R}$ (7)

**Remember**: *The circular frequency " **ω** " of current is defined by the source and is the same as that of its voltage.*

The graphs in figure 3b show the evolution of current and voltage across resistor with time and the figure 3a shows their *phasors* at a moment "*t*" in a *phasor diagram*. Those two presentations show that " ***i(t)*** *and* ***υ(t)*** *pass simultaneously through their maximal, minimal and zero values* ".

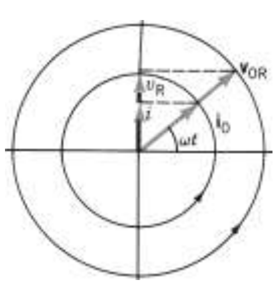
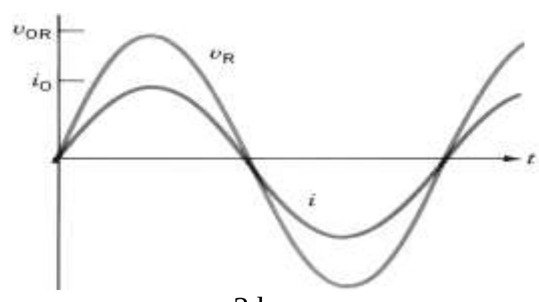



Figure 3.a




3.b

- What is the ***power dissipated as heat in resistor***? One may tent to find the average of current " ***i(t)*** " during one period and calculate an average power by formulae $P = (i_{av})^2 * R$ . But the graph of current in fig.3b shows that this $i_{av} = 0$ and one ***would get*** $P = 0$. Meanwhile, *with the exception of only two moments,* $i^2 > 0$ and the instantaneous dissipated power in resistor is $p(t) = i(t)^2 * R > 0$ which means that the ***average power dissipated*** in resistor *for one period must be* ***positive;*** i.e. it *cannot be zero. This is a good example which shows why each mathematical result has to be considered carefully if it does make sense or not from the physics point of view.*

For this calculation, one must start from *instantaneous power* $p(t) = i^2 R = i_0^2 R \sin^2(\omega t)$ (8)

Next, $P_{av} = \frac{1}{T}\int_0^T p(t)dt = \frac{1}{T}\int_0^T i_0^2 R \sin^2 \omega t dt = \frac{i_0^2 R}{T}\int_0^T \frac{(1-\cos 2\omega t)}{2} dt = \frac{i_0^2 R}{2T}\int_0^T dt = \frac{i_0^2 R}{2T} t \Big|_0^T = \frac{i_0^2 R}{2}$ (9)

**Notes:** a**)** One used trigonometric equity $sin^2\theta = \frac{1}{2}(1 - cos2\theta)$

b) $\frac{1}{T}\int_0^T \cos(2\omega t)dt = 0$ as *period average* for a harmonic function.

As the *average* ***of current square*** in a period is $\left(i^2\right)_{av} = \frac{i_0^2}{2}$ (10)

(*refer to integral (9) without R*) it comes out that $P_{av} = \left(i^2\right)_{av} R$ (11)

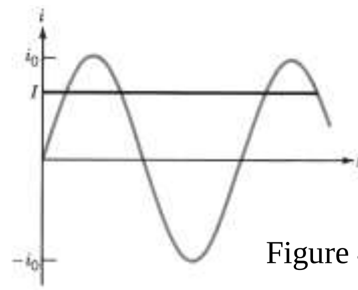



Figure 4

- One prefers to use for AC circuits similar formulas to those found for DC circuits. So, one has introduced the ***root mean square (rms)*** *current* (***I*** in fig.4) as $I_{rms} = \sqrt{(i^2)_{av}} = \frac{i_0}{\sqrt{2}} = 0.707 i_0$ (12)

Then, the *average power dissipated in resistor* $P_{av}$ (mostly called ***rms power***) is $P_{rms} = I_{rms}^2 R$ (13)

***Root mean square (rms)*** *potential difference* is defined as $V_{rms} = \sqrt{(v^2)_{av}} = \frac{v_0}{\sqrt{2}} = 0.707 v_0$ (14)

With this definition, one can write the Ohm's law in AC circuits as $V_R = IR$ (15)

- Expressions 12–15 relate the **rms** or ***effective*** values of ***current, voltage*** *and* ***power*** in an AC circuit. The ***effective power*** (*rms power at expression 13) is known also as* ***heating power*** of AC in a circuit. This amount of *power is dissipated as heat in a resistor* ***R*** and it is the same as that dissipated in this resistor by a DC current with $I = I_{rms}$. Note that the voltage **120V** provided by wall plugs in North America is a ***rms voltage***; it corresponds to a AC voltage with amplitude $v_o = 120/0.707 = 169V$. All devices that measure ***I, V*** in ***AC circuits*** provide ***approximate rms*** values. One must use devices labeled "*TRUE RMS values*" to measure AC parameters with a ***good precision***. <u>*The best way to measure all characteristics of AC is by use of a scope.*</u>

## 13.3 CAPACITOR IN AC CIRCUIT

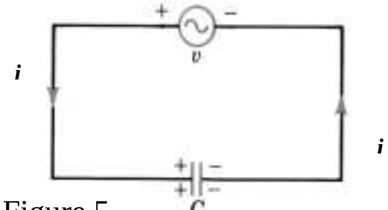



Figure 5

- The scheme in figure 5 shows a capacitor C connected "*in series*" to an ***AC*** source. Let's consider an interval ($t_o$, t) during which the charge ***q*** on C plates is ***increasing*** due to a "positive" *current in the circuit* that brings (+) charges to positive plate(see fig.5). In these circumstances one gets $\frac{dq}{dt} > 0$ and $\frac{dq}{dt} = i$ (16)

Remembering that the current evolution in time is assumed with a *phase constant* zero; $i = i_0 \sin(\omega t)$

one gets $\frac{dq}{dt} = i_0 \sin \omega t$ (17) So, $dq = i_0 \sin(\omega t)dt \rightarrow q = \int_{t_0}^{t} dq = \int_{t_0}^{t} i_0 \sin(\omega t)dt = \frac{i_0}{\omega}\int_{\omega t_0}^{\omega t} \sin x dx$

which gives $q = -\frac{i_0}{\omega}\cos(\omega t) + const(t_o)$ (18) One takes *const(t$_o$)* = 0 and this way gets rid of it.

So, the *evolution of charge* in capacitor is given by expression $q = -\frac{i_0}{\omega}\cos(\omega t)$ (19)

Then, the instantaneous difference of potential across capacitor plates is

$$\upsilon_C = \frac{q}{C} = -\frac{i_0}{\omega C}\cos(\omega t) \equiv -\upsilon_{0C}\cos(\omega t) \qquad (20)$$

**<u>Notes</u>**: **a)** The maximum *value of potential difference* across capacitor plates is $\upsilon_{0C} = i_0 \frac{1}{\omega C}$ (21)

**b)** As $i = i_0 \sin(\omega t)$ and $\upsilon_C = -\upsilon_{0C}\cos(\omega t) = \upsilon_{0C}\sin(\omega t - \pi/2)$, it comes out that the *potential difference at capacitor plates* is ***delayed (or lags) by π/2*** (see fig.6a.b) with respect to *current in circuit*.

Figure 6.a

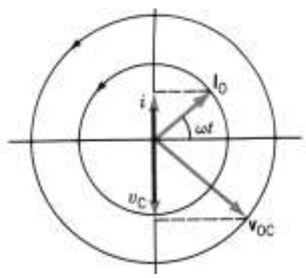
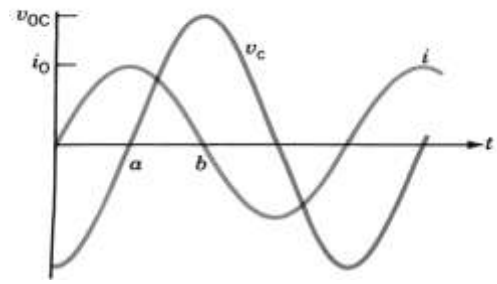



Figure 6.b




**c)** By applying the second rule of Kirchhoff for the loop; $\upsilon - \upsilon_C = 0$; so, at each moment $\upsilon_C = \upsilon$ (22)
The difference of potential at capacitor plates is equal to source voltage at each instant. At time " *t = a* " (fig.6.b) $\upsilon_C = \upsilon = 0 \Rightarrow\Rightarrow q = \upsilon_C * C = 0$. During the time (*a*, *b*), the potential difference $\upsilon_C$ (*and q* ) of capacitor increases while its *charging rate* " *dq/dt* " , i.e. the ***current*** in circuit, ***decreases***. At " *t = b* " when the voltage $\upsilon_c = \upsilon_{0C}$ , the current in circuit becomes zero. For " *t > b* ", the voltage $\upsilon_c$ of capacitor decreases (*due to oscillation of source voltage*). This means that the "+" charge starts to flow back from the capacitor into the source and the *current direction is inverted* (negative values of current in graph).

- So that one can use similar mathematical expression as in case of DC current, one has introduced the ***reactance of a capacitor*** $X_C = \frac{1}{\omega C}$ (23)

With this definition, the relation (21) can be rewritten as $\upsilon_{0C} = i_0 X_C$ (24)

As $i_0 = \frac{1}{0.707} * I\_and\_\upsilon_0 = \frac{1}{0.707} * V$ from (24) one gets $V_C = IX_C$ (25)

Note that " $X_C$ " in (24) relates the ***maximum* values** while in (25) it relates the ***rms*** (***effective***) values of *current in circuit* and *difference of potential on capacitor plates*.

- The expressions (24, 25) can be considered as applications of Ohm's law in AC circuit for an "***element with resistance = reactance*** $X_c$ ". The *reactance produces an infinite resistance* for DC current because for f = 0, ω = 0 one gets $X_C \sim (1/\omega = 1/0) = \infty$ . In a DC circuit, *after getting fully charged*, a capacitor interrupts the current through wires connected at its terminals. In a AC circuit its " *resistance* " becomes negligible for very high frequencies because $X_C \sim (1/\omega) = (1/\infty) \cong 0$; i.e. it acts as a "*short circuit* ". The increase of C-value ***decreases*** the ***reactance*** or the ***effective resistance*** in circuit because $X_C \sim (1/C)$.

## 13.4 INDUCTOR IN AC CIRCUIT

- The scheme in fig.7 shows an inductor L *in series* to an ***AC source***. Let's consider a moment when the *current* "***i*** " is *increasing* and it is directed as shown. The inducted ***emf*** ( $\varepsilon_L \equiv \upsilon_L$ *i.e. L-voltage*) builds up an induced current in opposite direction to "***i*** " in fig. By applying the second Kirchhoff rule for the circuit

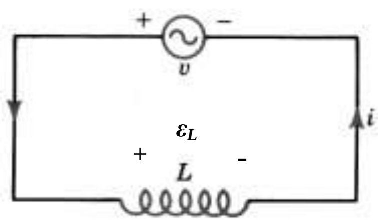



Figure 7

one gets $$\upsilon - \upsilon_L = 0 \qquad (26)$$
The *magnitude* of potential difference at inductor's terminals is
$$\upsilon_L = \varepsilon_L = L\frac{di}{dt} \qquad (27)$$
Relations (26, 27) define completely (*magnitude and direction*) the potential difference at inductor terminals. In the following we refer

to the voltage $\boldsymbol{\upsilon_L}$ (27) at inductor terminals instead of $\varepsilon_L = -L\dfrac{di}{dt}$

which negative sign is transferred in front of voltage $\boldsymbol{\upsilon_L}$ at relation (26).

- As the current in circuit at any moment is given by expression $$i = i_0 \sin(\omega t) \qquad (28)$$

$\dfrac{di}{dt} = i_0\omega\cos(\omega t)\_and\_\upsilon_L = Li_0\omega\cos(\omega t)$. So, voltage at inductor is $$\upsilon_L = \upsilon_{0L}\cos(\omega t) \qquad (29)$$

where $$\upsilon_{0L} = i_0\omega L \qquad (30)$$

Finally, one rewrites (29) in form $$\upsilon_L = \upsilon_{0L}\cos(\omega t) = \upsilon_{0L}\sin(\omega t + \pi/2) \qquad (31)$$

The *phasors* in figure 8.a and the graphs of expressions (28,31) in figure 8.b show that the ***voltage phasor at inductor*** terminals ***leads*** the ***current*** phasor into the inductor ***by π/2*** radians.

Figure 8.a

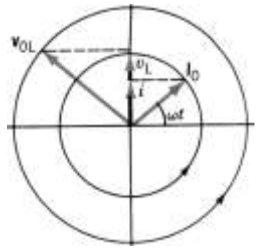
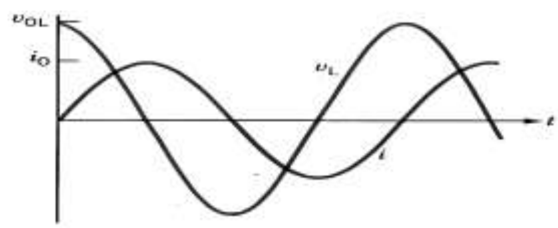



Figure 8.b




- To use the Ohm's law for an inductor in AC circuits, one has introduced the parameter $X_L = \omega L$ (32)
This parameter is known as the ***reactance*** and it measures the *resistance* of ***ideal inductor*** *in AC circuits*.
Then, the relation (30) for max. values can be rewritten as $$\upsilon_{0L} = i_0 X_L \qquad (33)$$
After multiplying both sides by 0.707, one gets $$V_L = I_L X_L \qquad (34)$$
The relations (33,34) relate the ***amplitudes*** *and* ***rms* values** *of voltage and current for an ideal inductor*.

- The relation (32) shows that, in contrary to capacitor, the resistance of an ideal inductor in an AC circuit (*i.e. the reactance*) increases ***with frequency*** increase. An *ideal inductor* would allow passing a DC current with *zero resistance* but it does present "*resistance"* to AC current. This "resistance" increases with the increase of frequency and the increase of its inductance (i.e. L-value).

**In AC circuits: - the Ohms law does apply only for max. values and rms values of current and voltage but it is *<u>not valid for instantaneous values</u>*.**
**- the resistance of a resistor " R " does not depend on frequency.**
**- the reactance of a capacitor " $X_C = 1 / \omega C$ " *decreases* with *frequency increase.***
**- the reactance of an inductor " $X_L = \omega L$" *increases* with *frequency increase.***