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3.1 Electrostatic Potential Energy, Potential Difference and Potential

REMEMBER from mechanics:

- *The potential energy can be defined only if there is* ***conservative forces*** *between objects of a system.*
- *Conservative forces* ***do not depend on velocity or acceleration****; they may* ***depend only on the position****.*
- *Only the difference between two values of potential energy (not the values by themselves) does have a physical meaning. This allows to assign zero potential energy to such a location that simplifies the calculations. In general, one chooses* $U_G = 0$ *for gravitational energy of an object on earth surface but if the object can move only inside the lab one chooses* $U_G = 0$ *at floor level. Note that one may choose* $U_G = 0$ *at the center of earth (or even at infinity) if this choice simplifies the solution of considered problem.*
- *The principle of mechanic energy conservation;* $\Delta E=\Delta U_G + \Delta K = W_{ext}$ *tells that, provided that* $\Delta K=0$*, the change of potential energy* ΔU_G *is equal to the work done by external forces.*
- *Consider an object at rest on floor where* $U_G(0) = 0$*. If you move it up at constant speed and leave on a table at height* y (fig.1)*, the object gets a potential energy* $U_G(y) = mgy$*. The external force*[1] *(your force) has done a* ***positive*** *work* W_{ext} *which increases the potential energy of object (i.e. of system object- earth).*

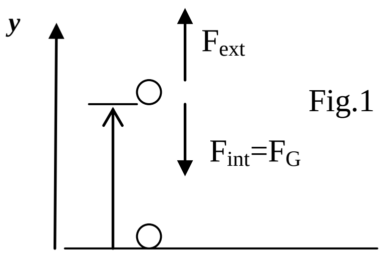

Fig.1

$$W_{ext} = \Delta U_G = U_G(y) - U_G(0) = U_G(y) \qquad (1)$$

Meanwhile, the ***internal conservative force*** F_{int} *(i.e. the weight), does the same amount of work but this one is a* ***negative work****. So,*

$$W_{int} = -W_{ext} = -\Delta U_G = -U_G \qquad (2)$$

- *The relation between the work by a conservative (internal) force and its effect on potential energy is*

$$\Delta U = -W_{int} \quad and \quad if \quad U_{in} = 0 \quad then \quad U = -W_{int} \qquad (3)$$

*For an infinitesimal displacement "****dy****", the internal force* F_{int}*, does the infinitesimal work*

$$dW_{int} = F_{int} * dy = -dU \qquad (4)$$

The sign (-) at relation (3-4) tells that if a ***conservative force*** *does* ***positive*** *work the* ***potential energy decreases****.*

From (4)one get the relation between a ***conservative force*** *and the potential energy* $F_{int} = -\frac{dU}{dy}$ (5)

Ex: Earth-object sys; If Oy directed vertically up $U_G(y) = mgy$ *,* $dW_{int} = F_{int} * dy = -dU_G$ *and internal force of system i.e. the gravitation force is* $F_{int} = F_G = -\frac{dU_G}{dy} = -mg$ *(i.e. directed down)*

- The potential energy "U_G" belongs to the *system earth-object* and it is due to the action of the earth **gravitational field**. One defines the ***potential function*** *or simply the* ***potential*** *of gravitational field* as follows: The gravitational **potential** " $V_G(y) = U(y)_G /m$ " is the **potential energy per 1Kg mass** placed at a **" y - location" of space.** So, $V_G(y) = mgy /m = g*y$ In SI system it has the **unit [J/kg]**.

This way, one:

a) defines ***gravitational potential*** $V_G(y) = g * y$ as a *characteristic* of *gravitational field* of the earth.

b) can get **potential** $V_G(y)$ and its change from the relations $V_G = \frac{U_G}{m}$ and $\Delta V_G = \frac{\Delta U_G}{m}$ (6)

c) can express the *work done by* gravitational *field* during the shift of a mass "m" inside it via V_G as

$$W_{int} = -\Delta U_G = -m * \Delta V_G \qquad (7)$$

[1] With respect to the system earth - object

- At this point we have to remind that the **Coulomb's force** is a **conservative force**[2], too. So, we adopt all the upper steps to the **electric field** and define the ***electric potential*** **V(x,y,z)** by expression (8). So

a) ***"V "*** represents the **potential energy** of "+***1C charge"*** at location (x,y,z) inside ***electrostatic field***.
b) Similarly to (1-7) we get its relations to W_{in} i.e. **work** done **by electric field** W_{el} on charge "q" as :

$$V = \frac{U_{el}}{q} = -\frac{W_{el}^{from\_"0"\ energy\ location"}}{q} \qquad (8) \qquad\qquad \Delta V = \frac{\Delta U_{el}}{q} = -\frac{W_{int}}{q} = -\frac{W_{el}}{q} \qquad (9)$$

From (8): The ***electric potential*** is equal to the ***"-" work done by electric field*** when "q = +1C" charge moves from the point with **0-potential energy** to the considered space location; ***its SI unit is [J/C≡V(volt)].***
From (9); The **change** of electric **potential** between any **two** space **locations** is equal to **" - " amount of work** **done by electric field** when a charge **+1C** gets **shifted from the initial to the final location** .

- When the electric field[3] shifts a ***positive*** *charge* along its lines, it achieves a positive work, $W_{el} > 0$. The relation (9) shows that $\Delta V = V_2 - V_1 = -(W_{el}/q) < 0$, which means that $V_2 < V_1$; the charge is shifted *from higher to lower potential*. So, if a **positive charge** is placed inside an electric field, the electric field tents to shift it versus locations with lower V-values(fig2.b). ***A similar situation is met inside gravitational field; the potential energy of objects in free fall decreases while they approach earth(fig.2a)***. But, if one places a **negative charge** "-q" inside a E-field, the electric force will move it against field lines. As ($\vec{\Delta S} \uparrow\uparrow \vec{F}_{el}$), the field achieves *positive internal work* ($W_{el} > 0$), but in this case $\Delta V = V_2 - V_1 = -(W_{el}/-q) > 0$ and $\Delta V = V_2 - V_1 > 0$ or $V_2 > V_1$. This means that the electric field pushes the negative charges versus higher V values (fig 2.c).

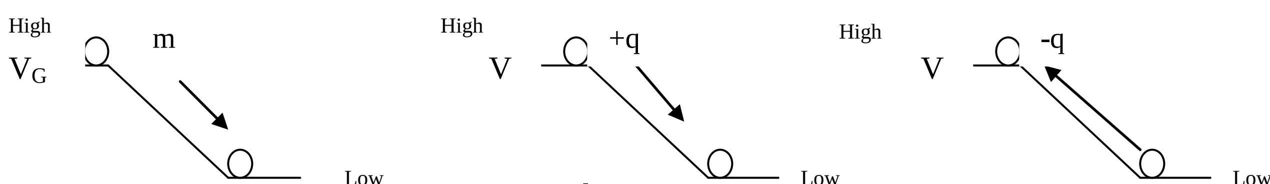

Fig.2a Fig. 2b Fig.2c

***The electric field** will **push** a **"+" charge** versus **lower V**-values and a **"-"** versus **higher V**-values.*

- If an "external action" shifts by $\vec{\Delta s}$ a charge " q " inside an electric field $\vec{E}$, the field does the work

$$W_{el.} = \overrightarrow{F_{el.}} * \overrightarrow{\Delta s} = q\vec{E} * \overrightarrow{\Delta s} = W_{int} = -q\Delta V \text{ (see 9)} \quad \text{which brings to} \quad \vec{E} * \overrightarrow{\Delta S} = -\Delta V \qquad (10)$$

For an infinitesimal shift one gets

$$\vec{E} * \overrightarrow{dS} = -dV \qquad (11)$$

or

$$dV = -\vec{E} * \vec{ds} = -E_s * ds \qquad (12) \qquad \text{and} \qquad E_s = -\frac{dV}{ds} \qquad (13)$$

The relations (9 -13) connect the two main parameters (i.e. E and V) of the same electric field.

-The electric **potential** is a sort of potential energy; so, **its value depends only on the location. This means that its change** depends only on the initial and final positions and *not on the path followed* for a shift from "A" to "B". This comment offers a guide for simplifying the calculation of work done by the electric field when a charge moves inside it, especially if it is a *non uniform* electric field (fig.3).
If a charge "q" moves inside a non uniform field, one calculates the *work done by the field* as follows:

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[2] Like universal gravitation law, the Coulomb's law depends only on the positions and not on velocities or acceleration.
[3] Internal conservative force in the system electric field - electric charge.

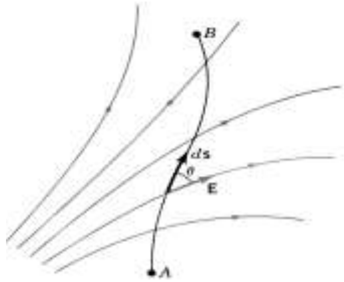


Figure 3

$$W_{el}^{A\to B} = \int_A^B q\vec{E} * \overrightarrow{ds} = q\int_A^B (\vec{E} * \overrightarrow{ds}) = q\int_A^B (-dV) = -q(V_B - V_A)$$

One writes this expression mainly in the form

$$V_B - V_A = -\int_A^B \vec{E} * \overrightarrow{ds} = -\frac{W_{el}^{A\to B}}{q} \qquad (14)$$

3.2 Potential and Potential Energy inside an Uniform Electric Field

-Consider that an *external force* shifts the charge "***q***" inside the *uniform electric field* $\vec{E}$ from A to B following an irregular path (Fig.4). One calculates the work done by electric force ($\vec{F}_{el} = q\vec{E}$) as

$$W_{el} = \int_A^{(path)B} \vec{F}_{el} * \overrightarrow{ds} = \int_A^{(path)B} q\vec{E} * \overrightarrow{ds} = q\vec{E}\int_A^{(path)B} \overrightarrow{ds} = q\vec{E} * \overrightarrow{\Delta s} \qquad (15)$$

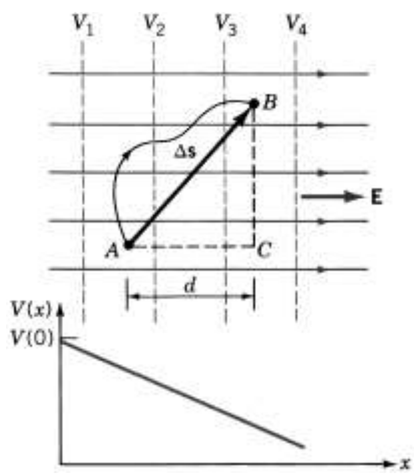


Fig. 4

(one can take out integral the vector $\vec{E}$ because it is constant)

and the sum of infinitesimal vector shifts is $\overrightarrow{\Delta s} = \overrightarrow{AB}$.

From (14,15) one get $V_B - V_A = -W_{el}/q = -\vec{E} * \overrightarrow{\Delta s}$

Then $$V_B - V_A = -\vec{E} * \overrightarrow{\Delta S} = -E * \Delta S_E = -E * d \qquad (16)$$

Note that the result is the same for any position of point B on the line CB. This means that for all these positions of point B the potential V_B has the same value. In other words:
The line CB is an equipotential line.

Next, one may fix the origin of an axis Ox at point A and select its direction along the field lines (fig.4) . As V=V(0) at origin, for a point on CB with coordinate ***x*** on Ox axis, relation (16) can be written in form:

$$V(x) - V(0) = -E * x \_and\_ V(x) = V(0) - E * x \qquad (17)$$

The relation (17) tells that the electric **potential decreases** linearly (Fig.4) **along** $\vec{E}$ **direction**.

One concludes that ***inside an uniform electric field***:

a) The **potential** value is the **same** on any plane **perpendicular** to electric **field** lines.
b) The **potential** ***decreases*** linearly while moving ***along*** the direction of **electric field**.
c) Dimensional rule applied at (16) gives [V]=[E]*[x] and [V]≡ ***volt*** **= [N/C]*[m]= N*m/*C* = *J/C***
Also, [E]= [V] / [x] i.e. another unit for electric field is ***[E]= V / m***

- So, we found that in the case of uniform electric fields :
THE ELECTRIC FIELD LINES ARE PERPENDICULAR TO THE EQUIPOTENTIALS AND POINT "DOWNHILL" FROM THE HIGHER TO THE LOWER VALUES OF POTENTIAL.
This is a general rule that applies for all electric fields (not only for uniform ones). Actually, if one shifts a charge q = +1C by $\vec{ds}$ *along* an equipotential line (on which ΔV = 0), the work done by the electric field is zero: $\vec{E} * \overrightarrow{ds} = W_{el} = -\Delta V = 0$. As $\vec{E} * \vec{ds} = 0$ means that $\vec{E} \perp \vec{ds}$ and $\vec{ds}$ is along an equipotential line, it comes out that the $\vec{E}$ ***- vector is perpendicular to equipotential lines***.

- Consider now a *free charge* "q" ***moving*** at *constant velocity* that enters inside an electric field. As no other forces apply on it, $W_{ext} \equiv 0$ and one can apply the principle of energy conservation for the *system electric field-charge "q"* ($\Delta E = W_{ext} = 0$). As long as this charge is inside the field its total energy will be constant ($\Delta E = \Delta K + \Delta U = 0$) and this means that, inside the electric field ΔK = - ΔU .
Then, as **ΔU = q* ΔV** one gets

$$\Delta K = -q * \Delta V \tag{18}$$

Depending on sign of its charge, the kinetic energy of particle may increase or decrease inside the field.
If **q > 0**, and the charged particle enters the field moving :
a) ↑↑ $\vec{E}$ *i.e. along* the direction of potential decrease ("*downhill*"), **ΔV < 0** and **the field will increase its K.**
b) ↑↓ $\vec{E}$ *i.e. along* the direction of potential increase ("*uphill*") , **ΔV > 0** and **the field will decrease its K**.
If **q < 0**, and the charged particle enters the field moving :
a) ↑↑ $\vec{E}$ *i.e. along* the direction of potential decrease ("*downhill*"), **ΔV < 0** and **the field will decrease its K.**
b) ↑↓ $\vec{E}$ *i.e. along* the direction of potential increase ("*uphill*"), **ΔV > 0** and **the field will increase its K**.

One has introduced a special ***unit*** "*electron volt or* **eV** " for the ***energy*** of microscopic particles. This unit is widely used in chemistry, atomic and nuclear physics. The electron- volt **is the kinetic energy gained by a particle with charge q = +e when moving *downhill* a difference of potential 1V.**

$$1eV = (1.602 * 10^{-19} C) * 1V = 1.602 * 10^{-19} J \tag{19}$$

3.3 The Electric Potential due to Point Charges

a) One point charge

- The electric field created by a point charge Q in the space around it is a *radial field*

$$\vec{E}(\vec{r}) = k\frac{Q}{r^2}\,\hat{r} = E(r)\hat{r} \tag{20}$$

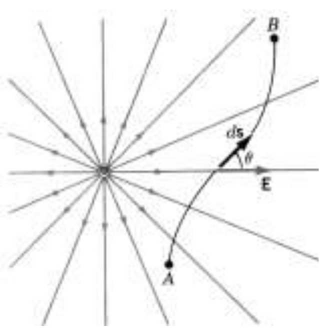


Fig.6
(Q > 0)

One may find the expression for the corresponding potential function by applying the expression (14) for a path AB inside this field (fig.6).

As $\vec{E} * \overrightarrow{ds} = E(r)ds * cos\theta = E(r)dr$ one get

$$V_B - V_A = -\int_A^B \vec{E}\,\vec{ds} = -\int_A^B E(r)dr = -\int_A^B k\frac{Q}{r^2}dr = -kQ\int_A^B \frac{dr}{r^2} = \\ = -kQ\left(-\frac{1}{r}\right)_A^B = kQ\left(\frac{1}{r_B} - \frac{1}{r_A}\right) \tag{21}$$

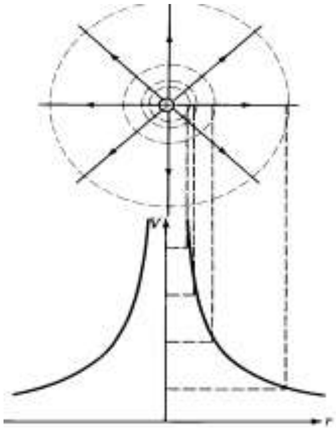


Fig.7 Electric potential due to a positive charge ($Q > 0$)

Next, one fixes the zero value of potential " at $r_A \rightarrow \infty$ ", i.e.$V(\infty) = 0$ and gets the potential expression at a point $r_B \equiv r$ as

$$V(r) - V(\infty) = V(r) = kQ\frac{1}{r} \qquad (22)$$

Figure 7 shows the profile of this function in 2D/3D space. The dashed circles are the equipotential lines. They get closer to each other near the charge +Q because the function V(r) changes rapidly in this region. The field lines are normal to equipotential surfaces. Note that the ***magnitude of field*** is ***larger*** at locations where the equipotential lines are ***denser*** (*near the charge which is the source of field*) and it decreases at locations where they are rare (*big r-values*).

b) Two point charges

- Consider now two point charges Q_1 and Q_2 at the same space region. Let $\vec{E}_1, \vec{E}_2$ be the electric field vectors due to each charge at a given point " B " at distance " r_1" from the charge Q_1 and " r_2 " from the charge Q_2. One can find the resultant electric field at point " B " by applying the superposition principle

$$\vec{E} = \vec{E}_1 + \vec{E}_2 \qquad (23)$$

As in previous section, one places the starting point "A" of a path "AB" at infinity (i.e. $V_A = V(\infty) = 0$) and express the potential difference as the sum of "$dV = -\vec{E} * \overrightarrow{ds}$ " terms during the shift from "A to B".

$$V_B - V_A = V_B - V_\infty = V_B = -\int_\infty^B \vec{E} * \overrightarrow{ds} = -\int_\infty^B (\vec{E}_1 + \vec{E}_2)\overrightarrow{ds} = -\int_\infty^B \vec{E}_1\overrightarrow{ds} - \int_\infty^B \vec{E}_2\overrightarrow{ds} =$$

$$= -\int_\infty^B E_1(r_1)dr_1 - \int_\infty^B E_2(r_2)dr_{12} = -\int_\infty^B k\frac{Q_1}{r_1^2}dr_1 - \int_\infty^B k\frac{Q_2}{r_2^2}dr_2 = \cdots = \frac{kQ_1}{r_{1-B}} + \frac{kQ_2}{r_{2-B}}$$

If the net field is due to several charges $Q_1, Q_2, ..Q_i$ the net potential is calculated through the expression

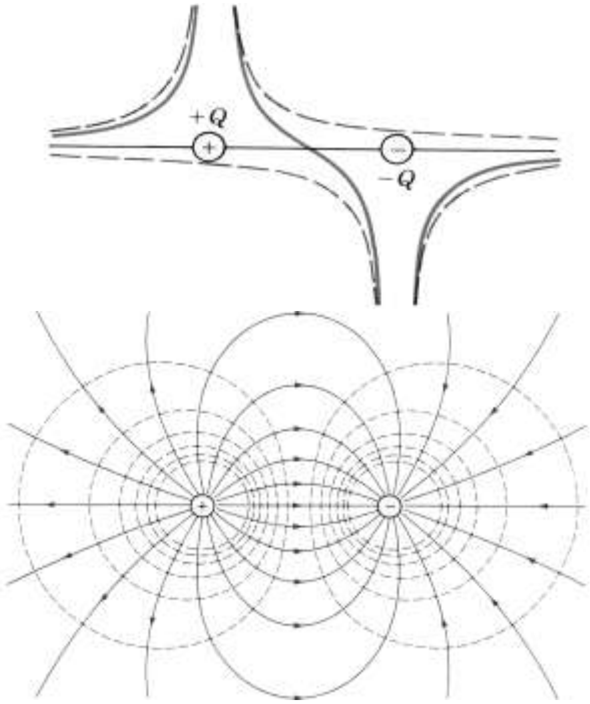
$$V_B = \sum_i \frac{kQ_i}{r_{i-B}} \qquad \text{or more generally as} \qquad V(r) = \sum_i \frac{kQ_i}{r_i} \qquad (24)$$

Notes: a) " $\boldsymbol{r_i}$ " is the ***distance*** from "***i-charge*** " to the location where is the point ("*B*") where one measures the value of electric potential.

b) one must take into account the **signs** of charges inside the sum (24).

-Figure 8.a *(up)* shows the *evolution of electric potential V* inside the electric field around a dipole; (*down*) it shows the *equipotential lines* and the electric *fields' lines due to an electric dipole*.

- Figure 8.b*(up)* shows the *of evolution of electric potential V* inside the electric field built by two equal positive charges; (*down*) it shows the *equipotential lines* and the field lines in the plan that contains electrical charges.



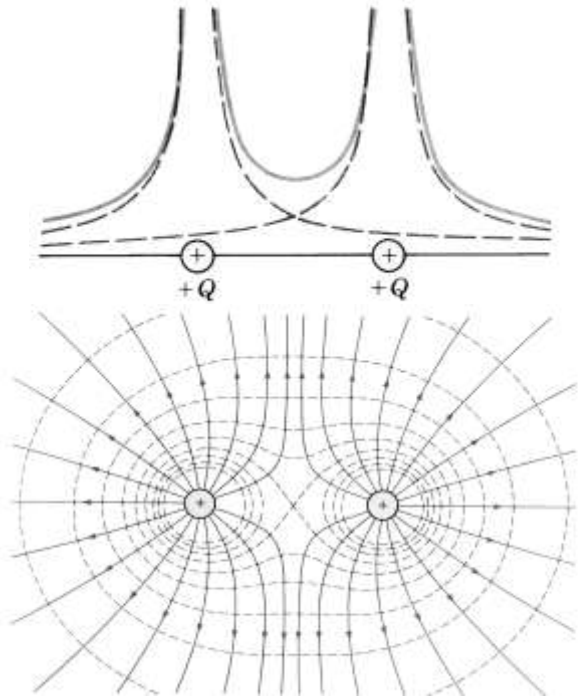


Fig. 8a

Fig.8b

3.4 General formula relating Electric Field to Electric Potential

- From the relation $dV = -\vec{E} * \overrightarrow{ds} = -E * ds * cos\theta = -E_s ds$ (25)

one can get $E_s = -\frac{dV}{ds}$ (26)

In those relations, $\boldsymbol{E_s}$ is the *component of field* along an arbitrary direction $\vec{ds}$ inside the field. This means that the component of $\vec{E}$ vector on a given direction is related to the change of " V " along same direction.

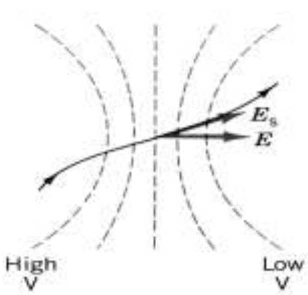


Fig.9

By introducing a rectangular system of axes Oxyz, one can get

$$\vec{ds} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k} \quad \text{and} \quad \vec{E} = E_x\,\vec{i} + E_y\,\vec{j} + E_z\,\vec{k} \qquad (27)$$

Then, $$dV = -\vec{E} * \vec{ds} = -(E_x dx + E_y dy + E_z dz) \qquad (28)$$

By considering a displacement in Ox direction, dy = dz = 0 one get $dV = -E_x dx$. So, it comes out that

$$E_x = -\left(\frac{\partial V}{\partial x}\right)_{y,z\_const} \quad \text{and} \quad \text{similarly, } E_y = -\left(\frac{\partial V}{\partial y}\right)_{x,z\_const} \quad \text{and } E_z = -\left(\frac{\partial V}{\partial z}\right)_{x,y\_const}$$

Then, from (27) one finds out the expression $\vec{E} = -\left(\frac{\partial V}{\partial x}\vec{i} + \frac{\partial V}{\partial y}\vec{j} + \frac{\partial V}{\partial z}\vec{k} \right)$ (29)

The expression $gradV(x,y,z) = \left(\frac{\partial V}{\partial x}\vec{i} + \frac{\partial V}{\partial y}\vec{j} + \frac{\partial V}{\partial z}\vec{k} \right)$ is known as **gradient of function V**(x,y,z).

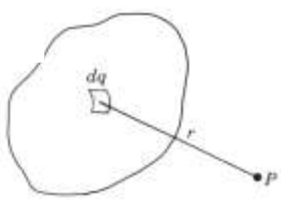
In many textbooks, the expression (29) is written in the form $\vec{E} = -gradV(x,y,z)$ (30)

3.5 Continuous Charge Distributions

-The potential, due to a continuous charge distribution, at a point P inside a field can be found by:

a) Expressing the contribution of each elementary charge " dq " as $dV = k\frac{dq}{r}$
b) Calculating the total potential as sum of elementary contributions over the total distribution as

Fig. 10



$$V = k \int_{region\_of\_charge} \frac{dq}{r} \quad (31)$$

3.6 Conductors inside an electrostatic field

- As explained in section 2.3, the ***electrostatic*** field is **zero inside** a conductor and **perpendicular** to its surface ***outside nearby***. As $V_B - V_A = -\int_A^B \vec{E} * \vec{ds} = 0$ for any path connecting **two points located inside it or on** its surface, it comes out that **the electric potential of a conductor is the same everywhere on its surface and inside it**. Note that this is *valid for any conductor(neutral or charged) placed inside an electrostatic field*.

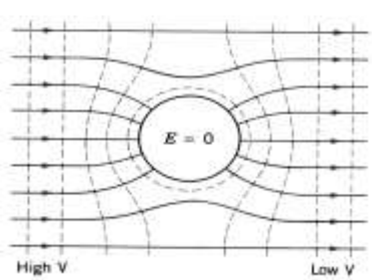


Fig.11

One might remember that that the presence of a conductor in an external field modifies the *field pattern* around its surface so that the *field lines* become perpendicular to its surface. Fig.11 shows the deformation of *equipotential planes* (and field lines) of a uniform electric field around a spherical uncharged conductor. Near the sphere, the equipotential planes become spherical surfaces.

-Also, at a charged *conductor* : the electric *charges are distributed on its surface*, the electric *field inside it is zero* and the potential is constant in all its volume and its surface. These rules remain the same if one open a cavity inside conductor and use the *conductor surface as a shield from external fields*. Faraday cage makes use of this shield effect to block the action of external fields in the space inside it.

-Assume that two conducting objects "1, 2" have charges "Q_1 , Q_2". Once one connects them by a wire, their charge will redistribute so that the electric **potential become the same** on any point on surface and inside the set(object_1, object_2 , wire). At "*electrostatic state*" the electric field vector $\vec{\boldsymbol{E}} = \boldsymbol{0}$ at any point inside each part of the set; outside, it is directed perpendicular to their surface.

-If one places a charge "Q" on a metallic sphere, it will get distributed uniformly on sphere surface and this will build a constant charge density on sphere surface $\sigma = \frac{Q}{4\pi R^2}$ (32)

It is clear that the charge density on surface is larger if radius R is smaller. If the surface of the metallic object is not spherical, the charge density is not uniform; it is higher at the at sharp points (*small R*) and lower at less curved (*large R*) areas. Consequently the magnitude of electric field outside metal is larger ($|\vec{E}| \sim \sigma$) in front of sharp points. One uses this effect to produce electric fields with large magnitude in restricted locations. Note that, this effect may become problematic because it ionizes the air nearby and may be the origin of corona discharges.

REMEMBER:

1) V(r) = -W_{field} (+1C shifted_from _infity _to_r) and $\Delta V = V(B) - V(A) = -\Delta W_{field}$((+1C shifted_from A→B).

One should not forget that V is not exactly a potential energy; its unit is "***volt =J/C*** " and not "***J* ".**

2) For a charge "q" the **potential energy** is U = q *V , ΔU = q*ΔV and $W_{field} = -\Delta U = -(q*\Delta V)$

3) A single point charge "Q" provides the potential $V(r) = kQ\frac{1}{r}$ in the space around it

and many point charges provide the potential $V(r) = \sum_i \frac{kQ_i}{r_i}$ at a given space location .

4) The change of electric potential between two points inside an electric field is $\Delta V = -\vec{E} * \overrightarrow{\Delta s}$.

5) $\vec{E}$ is always ***perpendicular to equipotentials (surfaces or lines)*** and ***<u>directed versus the decrease</u>*** of potential values.

$$\vec{E} = -\left(\frac{\partial V}{\partial x}\vec{i} + \frac{\partial V}{\partial y}\vec{j} + \frac{\partial V}{\partial z}\vec{k}\right)$$