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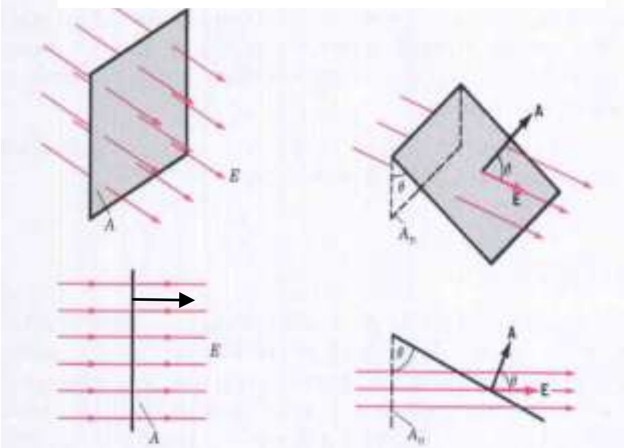
THE FLUX OF A VECTOR FIELD THROUGH A SURFACE

As introduced previously, the direction and the magnitude of electric field vector in any location of space is described by the set of field lines. The field lines are not only useful for visual and general information about the field but, via their "***flux"***, a specific physical parameter, they help to solve many problems, too.

- Assume that an uniform electric field is set in a space region and "A" is a ***flat surface*** inside this region. One introduces the "***area vector"*** $\vec{A} = A\hat{n}$ which is a vector with magnitude equal to numerical value of area A; its direction is fixed by a unit vector $\hat{n}$ perpendicular to surface. In case of Fig.1a, the plan A is perpendicular to $\vec{E}$ field and the vector $\vec{A}$ is directed along field lines but, in general, A-surface can have any direction " θ " versus $\vec{E}$ field (1.b). The **flux** of ***electric*** **field** through the area A is a scalar defined

Fig.1a Fig.1b



as $$\Phi_E = \vec{E} * \vec{A} \quad (1)$$

In the case 1.a , the angle between vectors is 0^o and Φ_E = EA. If the plan is not perpendicular to direction of field lines (see 1.b), the flux is

$$\Phi_E = E*A*cos\theta \quad (2)$$

where θ is the angle between the field lines and ***area vector***. In general, one selects $0^o < \theta < 90^o$.
The unit of electric flux in SI system is
$[\Phi_E]$ = (N/C)*m^2 or (V/m)*m^2 = V*m

Important note*: The presentation of E-field lines is a qualitative presentation that gives fast information about field. The flux Φ_E of $\vec{E}$ field through an area is a quantitative measurable parameter.*

- One can calculate the flux even if the field is not uniform and the area is not flat by dividing the area (fig.2) into small pieces $\boldsymbol{\Delta A_i}$ and assuming that the field is constant on each small area. Next, one applies expression (1) for each small area "***i***" $\Delta\Phi_i = \vec{E}_i * \overrightarrow{\Delta A}_i$ and get the total flux

through all surface as $$\Phi_{Net} = \Delta\Phi_1 + \Delta\Phi_2 + \cdots \Delta\Phi_n = \sum_{i=1}^{n} \vec{E}_i * \overrightarrow{\Delta A}_i \quad (3)$$

In the limit, the expression (3) transforms to $$\Phi_{Net} = \iint_{all\_area} \vec{E} * \overrightarrow{dA} \quad (4)$$

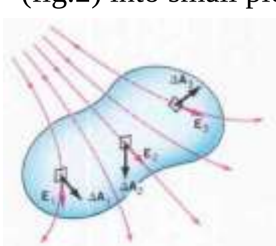


Fig.2

- The calculation of integral (4) is not difficult in the case of *symmetrical fields and closed areas*. For ***closed areas***, one applies the following rule:
All the *area vectors* are *normal to* the *surface* and are *directed outward (fig.3).*

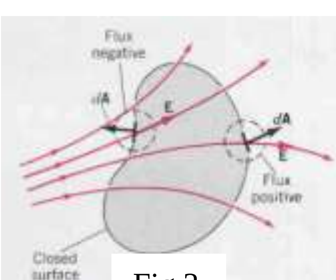


Fig.3

One should keep in mind that the field lines that ***enter into*** a closed surface have $90^o < \theta < 180^o$, i.e. ***cosθ < 0*** and produce a negative flux. The field lines that ***go out*** a closed surface have $0^o < \theta < 90^o$, i.e. ***cosθ > 0*** and produce a positive flux.
After calculating the algebraic value of flux due to each particular component, one find the net flux by taking their sum (3 or 4).

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THE GAUSS LAW

The Gauss law states that the total flux of an electric field through a closed surface is equal to the total (net) electric charge enclosed inside this surface divided by ε_0. $\Phi_E = \oiint_{closed\_area} \vec{E} * \overrightarrow{dA} = \frac{Q_{inside}}{\varepsilon_o}$ (5)

- Ex.#1. ***Electric charge +Q distributed uniformly through the volume of a sphere with radius R***.

1a. *Electric field **outside** the sphere*. From symmetry considerations, one figures out that electric field has a radial direction. Also, due to symmetry, at a given distance r(>R) the field vector $\vec{E}$ has the same magnitude. So, one may build a Gauss sphere (Fig.4) with radius "***r***" and apply the relation (5) on it. As, at any point on this surface $\vec{E} \uparrow\uparrow \overrightarrow{dA}$, it comes out that

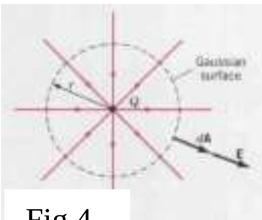

Fig.4

$$\Phi_E = \oiint_{sphere} E * dA = E \oiint_{sphere} dA = E * 4\pi r^2 = Q / \varepsilon_0$$

and one get $E(r) = \frac{Q}{4\pi\varepsilon_0 r^2}$ which is the same as predicted by Coulomb's law.

By the same steps, one gets the expression $E(R) = \frac{Q}{4\pi\varepsilon_0 R^2}$ for E-field magnitude **on** the sphere **surface**.

1b. *Electric field inside the sphere*. If r < R , one can use a Gauss sphere with radius "r" .From symmetry considerations, one can figure out that the $\vec{E}$-field is radial, directed outside and has equal magnitude everywhere on sphere surface. Similar calculation as in case (1a.) give $E(r) = \frac{Q(r)}{4\pi\varepsilon_0 r^2}$ where Q(r) is the part of charge inside sphere with radius r. As there is uniform charge distribution in volume, one labels the volume charge density by "ρ" and express the sphere charge, depending on its radius, as $Q = \frac{4}{3}\pi R^3 * \rho$ and $Q(r) = \frac{4}{3}\pi r^3 * \rho$. By taking the ratio of these two expressions one get $\frac{Q(r)}{Q} = \frac{\frac{4}{3}\pi r^3 * \rho}{\frac{4}{3}\pi R^3 * \rho} = \frac{r^3}{R^3}$ which brings to $Q(r) = Q\frac{r^3}{R^3}$ and finally to $E(r) = \frac{Q}{4\pi\epsilon_0 R^3} * r$. Figure 5 shows the evolution of magnitude vector $\vec{E}(r)$ for locations " r " inside and outside the sphere. ***Note that E=0 at sphere center***. This result confirms that a *point charge* builds a 0- field *at its own location*.

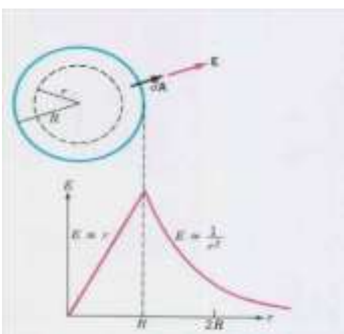

Fig.5

Ex.#2. ***Electric charge +Q distributed uniformly with linear density λ on an infinite long thin wire***.

From symmetry considerations, one figures out that the electric field has the same magnitude at all points at distance "r" from the wire and the electric field vector is directed from the wire "outside". Next, one considers a *cylindrical Gauss surface* with base radius "r" and side length "L". The direction of electric field on the two basis is ⊥ to area vector. As $\cos 90^0 = 0$ the flux on two basis is zero. So, the total flux is due to sides where $\cos 0^0 = 1$ and $\Phi = (2\pi rL) * E$. The formula (5) gives $\Phi_E = (2\pi rL)E(r) = \frac{Q}{\epsilon_0} = \frac{\lambda L}{\epsilon_0}$ and $E(r) = \frac{\lambda L}{(2\pi rL)\epsilon_0} = \frac{\lambda}{(2\pi\epsilon_0)r}$ or $E(r) = \frac{2\lambda}{(4\pi\epsilon_0)r} = \frac{2k\lambda}{r}$ which is a formula we have derived previously.

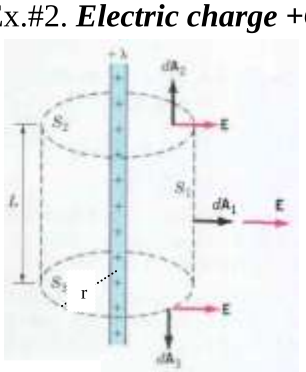

Fig.6

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Ex.#3.a ***Electric charge +Q distributed uniformly at surface density "σ " over an infinite flat plan***.

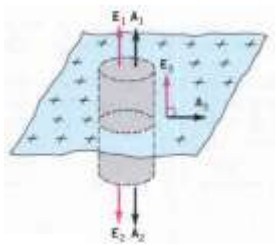

Fig.7

From symmetry considerations, one can figure out that the electric field has the same magnitude at all points at same distance from the surface and the electric field vector is directed perpendicular to plane. Next, one considers a cylindrical Gauss surface as shown. The direction of electric field on the side of cylinder is perpendicular to area vector. As $\cos 90^0 = 0$ the flux on the side area is zero. So, the total flux is due to two circular basis where $\cos 0^0=1$ and $\Phi = 2(A*E)$. Then, $\Phi_E = 2AE = Q/\varepsilon_0 = \sigma A / \varepsilon_0$ and $\boldsymbol{E = \sigma / 2\varepsilon_0}$ which is a formula derived previously.

3b. ***Electric charge +Q distributed uniformly with surface density "σ" on an infinite conducting plate***.

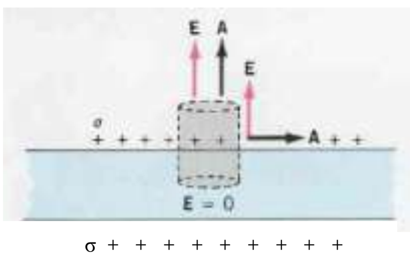

Fig.8

In the case of a single conducting ***plate*** the charge is distributed over the two opposite side surfaces equally; that means the same surface charge density. This distribution produces zero electric field inside the volume of plate . By applying the Gauss law over the cylinder shown in figure 8, one gets $\Phi_E = AE = Q/\varepsilon_0 = \sigma A / \varepsilon_0$ and $E = \sigma / \varepsilon_0$ which is twice larger than the field from a charged flat ***plane***. This does make sense because charges on both sides of plate contribute in the same direction to the field outside the plate. A similar calculation shows that the electric field has the same magnitude $E = \sigma / \varepsilon_0$, close to the other side of plate .

3.c In the case of a charged capacitor (fig.9), all charges in excess that come from the source are ***located only on the inner side of each plate*** because they are attracted from the opposite sign charges located on the other plate of capacitor. Each of these charged layers acts as a thin *flat plane* and produces an electric field with magnitude $E = \sigma / 2\varepsilon_0$ in the region between plates. Both these fields have the same direction (from "+" to "-" charges) and the same magnitude; so the resultant field is directed from "+" to "-" charges and has a magnitude $\boldsymbol{\sigma/\varepsilon_0}$. Note that inside the volume of each plate, the electric field is zero because otherwise it would produce motion of e- of metal plate. By using a small Gauss cylinder (see fig.9) with base area "A" that contains charge "Q=σA" and has E_{up} over the upper base, one gets $\Phi_E = AE_{up} + A\,{}^{\sigma}/_{\varepsilon_0} = {}^{Q}/_{\varepsilon_0} = {}^{A\sigma}/_{\varepsilon_0}$ and finally $E_{up} = 0$. ***So, the electric field is localized only in the region between the two capacitor plates***.

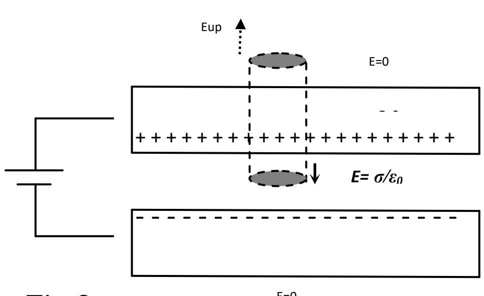

Fig.9

Ex.#4. Electric field close to the surface of a conductor

If an excess charge is placed on a conductor, it will get distributed on its surface. If the surface is a sphere the charge get distributed uniformly at constant surface density. In other cases the surface density will not be constant and will depend on location. As explained previously, there is zero field inside a conductor and the external field is perpendicular to its surface.

By using a small Gauss cylinder with one tap end inside and one outside (*like in fig.8*) one may find that $\Phi_E = AE = Q/\varepsilon_0 = \sigma A / \varepsilon_0$ and $\boldsymbol{E = \sigma / \varepsilon_0}$ where ***σ is the local charge density***. So, ***the electric field outside and close to the surface of a conductor is proportional to its surface density of charges at this location***.