1) SIMPLE HARMONIC MOTION/OSCILLATIONS

Introduction:

- One says that an "*event* " is ***periodic*** if it is repeated at regular intervals in *space* or in *time*.

Periodicity in Space is the regular appearance of same "*object*" or "*feature*" in *space (ex: the "cm lines" on a ruler; the diffracting slits in an optical grating; the atoms in a crystalline structure of a solid).*

Periodicity in Time is the recurrence of the "*same event*" after equal *time* intervals (*ex:the same* ***position and velocity*** *of the ´block´ in a block-spring system or the ´bob´ in a pendulum or piston in a car engine).*

- During a *periodic motion*, the object oscillates regularly around a certain *equilibrium location* from which one calculates the *displacement*. The graph of ***displacement*** vs. ***time*** *contains all the information* about the details of an oscillatory motion.

By definition, if an object's ***displacement*** from equilibrium follows a **pure harmonic** (sine or cosine) ***function of time***(fig.1), one says that it is performing a Simple Harmonic Motion (SHM). Experiments show that this happens in the case of a block-spring or a simple pendulum motion " *without friction* ".

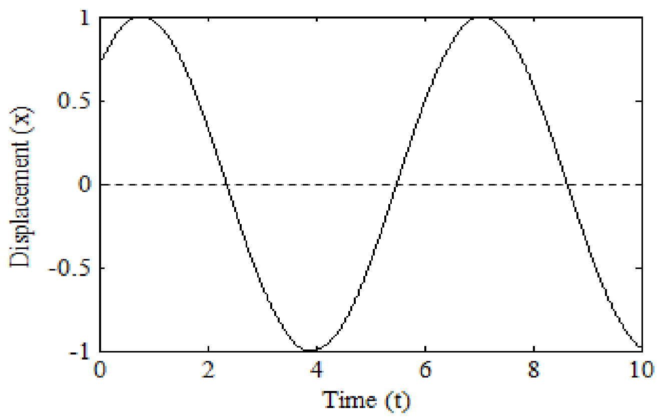
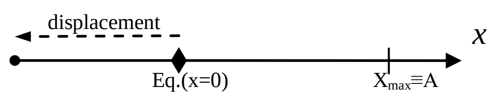


Figure 1a: The harmonic graph of displacement as a function of time in a SHM. Depending on the time, the *displacement* of object versus its equilibrium location (shown by the dashed line in graph) may be either positive or negative.

Figure. 1b: Displacement of CM of block in a horizontal block spring system



Phasor, Functions & Equation of SHM

In a SHM, the ***displacement*** is a ***harmonic function*** of ***time;*** so, it should have the form $y = A\sin\varphi(t)$. One uses the ***phasor model*** (fig. 2) to get the form of expression "$\varphi(t)$" at harmonic function. The ***phasor*** is a ***vector*** with magnitude "A" that rotates at a ***constant angular frequency*** "ω" while its tail is fixed at the origin of Oxy frame. The ***angle*** "φ" to Ox axe of phasor shows the ***phase.*** The values of "ω , φ" are taken positive for *contra-clockwise* (CCW) rotation and negative for *clockwise* (CW) rotation.

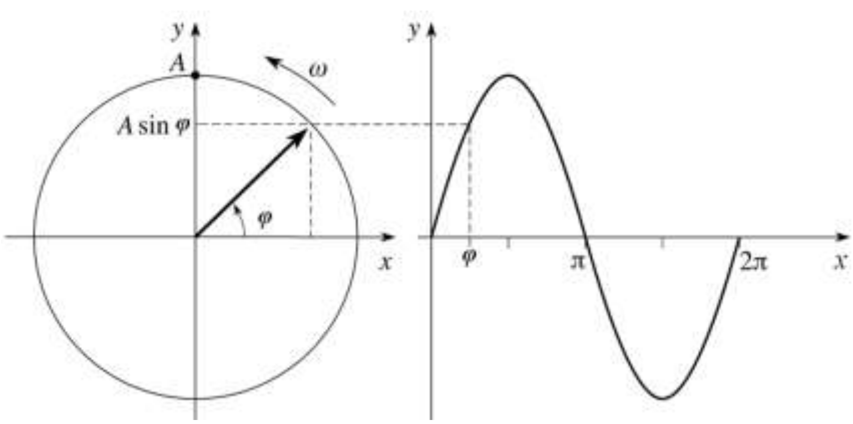


Figure 2: The phasor model allows to visualize the values of the phase and the displacement at any moment. If the angle (i.e. the ***phase***) of this vector with respect to the $+x$-axis is φ_0 at t = 0, then at time " t " it will be

$$\varphi(t) = \varphi_0 + \omega t \quad (1)$$

The magnitude "A" of the vector in a phasor model is taken equal to the maximum shift of CM of block from its equilibrium position. This parameter is known as the ***amplitude*** of oscillations. With the use of trigonometry, one may find that the *y- component* of phasor is $y = A\sin\varphi(t)$ (2)

Then, by substituting (1) into (2), one obtains the following harmonic function for *displacement* vs. time :

$y(t) = Asin(\varphi_0 + \omega t)$ which is mostly written as $\boldsymbol{y(t) = Asin(\omega t + \varphi_0)}$ (3)

One can describe a SHM by referring to the *x-component* of the phasor, too. In this case one would get

$x(t) = Acos(\omega t + \varphi_0)$. However, <u>in this course we will refer to the *sine* component</u>.

A vector with **amplitude *A*** (*maximum displacement*) rotating CCW at a constant "*angular frequency* $\boldsymbol{\omega}$" is known as a <u>*phasor model*</u>. It is the basic model used in physics for the study of oscillations and waves.

How to define the values of parameters A, ω, and φ_0 for oscillations of a given block-spring set ?
The *A*- parameter is equal to the maximum shift of block CM from equilibrium. To define ω - value, one uses the oscillation ***period*** " ***T*** ". During this time the phasor completes one full rotation(see fig. 2) . This means that, during the time " ***T*** " the phasor rotates by the angle 2π and from relation (1), one gets:

$\boldsymbol{\Delta\varphi(t) = \Delta\varphi_0 + \omega\Delta t = \omega\Delta t}$ (as $\Delta\varphi_0 = 0$), which for $\boldsymbol{\Delta\varphi(T) = 2\pi}$ and $\boldsymbol{\omega\Delta t = \omega T}$ gives $\boldsymbol{2\pi = \omega T}$ (4)

The <u>*real or **natural frequency***</u> of oscillations is defined as: $\boldsymbol{f = \frac{1}{T}}$ (5)

From (4) and (5), one gets the relationship between the ***circular*** "$\boldsymbol{\omega}$" and the ***natural*** "***f*** " frequencies:

$$\boxed{\boldsymbol{\omega = \frac{2\pi}{T} = 2\pi f}} \qquad (6)$$

Knowing the value of the displacement at t = 0, one may get $\boldsymbol{\varphi_0}$ as: $\varphi_0 = \arcsin\left[\frac{y_{(t=0)}}{A}\right]$. Generally, one prefers to use values between " 0 *and* ±2π " for the ***phase constant*** $\boldsymbol{\varphi_0}$. Note that using $\boldsymbol{\varphi_0} = 3\pi/2$ is the same as using $\boldsymbol{\varphi_0} = -\pi/2$. If the displacement of object from equilibrium position is zero at t=0, then the *phase constant* is zero or pi ($\boldsymbol{\varphi_0} = 0$ or $\boldsymbol{\varphi_0} = \pi$) ; otherwise it has a value between 0 and ±2π.

<u>Notations & Dimensions:</u>

y – Displacement (depending on time, it may be positive, negative, or zero);
A – Amplitude of oscillations (always positive);
$\boldsymbol{\varphi_0}$ – Phase constant (radians);
f – Natural frequency (Hz = 1/s)
$\boldsymbol{\varphi}$ – Phase (radians);
T – Period (seconds);
$\boldsymbol{\omega}$ – Circular frequency (rad/s).

A SHM is an oscillation with:

a) Constant amplitude and constant period(*see next sections*).
b) Harmonic (*sine* or *cosine*) time dependence.
c) Constant energy(*see next sections*).

- **Differential equation for SHM** $\boldsymbol{y}(\boldsymbol{t}) = A sin(\omega t + \varphi_0)$ (7)

$\Rightarrow \boldsymbol{y}'(\boldsymbol{t}) = A\omega \cos(\omega t + \varphi_0)$ (8)

$\Rightarrow \boldsymbol{y}''(\boldsymbol{t}) = -\omega^2 \underbrace{A sin(\omega t + \varphi_0)}_{\boldsymbol{y(t)}}$ (9)

The last relation can be presented as:

$$\boxed{\frac{d^2y}{dt^2} = -\omega^2 y} \quad OR \quad \boxed{\frac{d^2y}{dt^2} + \omega^2 y = 0} \tag{10}$$

The equation (10) is known as SHM **equation** ; the expression (7) is the SHM **function**.

Since ' ***y*** ' is the *displacement*, its second derivative ***y''*** gives the acceleration "***a*** ", i.e. ***y''(t)*** = ***a(t)***. Then, from relation (9) comes out that:

$$\boldsymbol{a(t) = -\omega^2 y(t)} \tag{11}$$

Let's take closer look at (9). The maximum possible value of $sin(\varphi_0 + \omega t) = 1$ (since $-1 \leq sin(x) \leq 1$). This means that the maximum value of acceleration is $a_{max} = A\omega^2$ (12)

Similarly, as the first derivative of the displacement, the relation (8) gives the velocity as a function of time. Also, as $-1 \leq cos(x) \leq 1$, it comes out that $\upsilon_{max} = A\omega$ (13)

Note: The relations (10) and (11) hold only for a SHM and can be used as a criterion for determining whether a given oscillation is a SHM or not.

Remember: $cos(x) = sin(x + \pi/2)$ and $-sin(x) = sin(x + \pi)$. Using these trigonometric identities, we may transform (8) into: $\boldsymbol{y'(t) = A\omega sin[(\omega t + \varphi_0) + \pi/2] = v(t)}$ (8')

and (9) into $\boldsymbol{y''(t) = A\omega^2 sin[(\omega t + \varphi_0) + \pi] = a(t)}$ (9')

Now, (8') and (9') are the *y-components* of two new phasors, the ***velocity phasor*** with ***magnitude Aω*** and the ***acceleration phasor*** with ***magnitude Aω²***. So, the three phasors (displacement, velocity, acceleration) rotate contraclockwise at same circular frequency "***ω***". From (8') one may notice that the velocity phasor is always ahead of the displacement phasor by π/2 rad. Also, from (9') we see that the acceleration phasor is π rad ahead of the displacement phasor (figure 3).

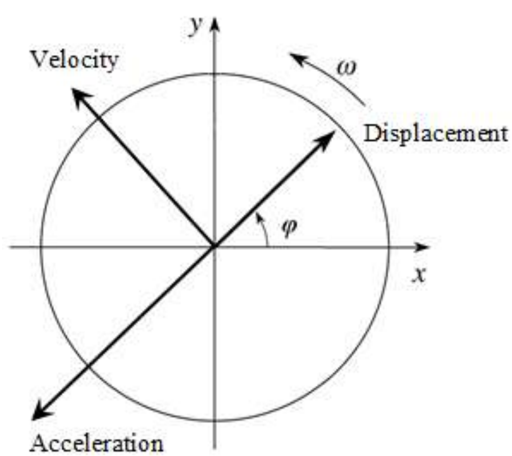


Figure 3: Three phasors with their corresponding phase shifts. The velocity phasor is π/2 *rad* ahead of the displacement phasor. The acceleration phasor is π *rad* ahead of the displacement phasor. All those three phasors rotate CCW simultaneously at the same angular frequecy *ω* (see video#1).

video#1 : http://laplace.us.es/wiki/index.php/Archivo:Fasorxva.gif

Horizontal Block-Spring System

Consider a block with mass ***m*** tied to the end of an extended spring *without mass*. Assume that the block is sliding *without friction* on a horizontal plane. In these circumstances, the gravitation force is cancelled by normal force and the "moving force" applied on the block is equal to the spring force exerted on it (fig.4). The spring extension is equal to the displacement "x" of the block CM from the equilibrium position. The spring's force on block is equal to elastic force of spring which is given by Hooke's law

$$F_x = F_{sp}^{el} = -kx \qquad (14)$$

where k is the elasticity constant of the spring.

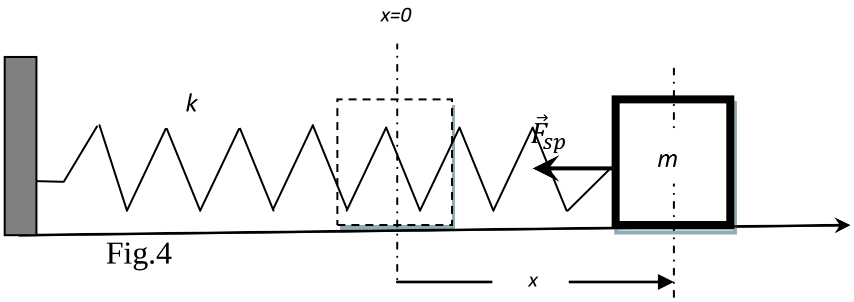


Fig.4

By applying the Newton's second law, along x-direction, we get:

$$\boldsymbol{F_x = ma_x} \qquad (15)$$

Next, by the use of (14) :

$$F_x = \boldsymbol{ma_x} = \boldsymbol{-kx} \ldots \boldsymbol{a_x} = -\frac{\boldsymbol{k}}{\boldsymbol{m}}\boldsymbol{x} \qquad (16)$$

As ***k*** and ***m*** are positive quantities, one can assign

$$\boldsymbol{\omega^2} = \frac{\boldsymbol{k}}{\boldsymbol{m}} \qquad (17)$$

and the relation (16) takes the form $\boldsymbol{a_x = -\omega^2 x}$. This relation can be written as $\frac{d^2x}{dt^2} = -\omega^2 x$ which is the SHM equation (10). One concludes that the block (or the system block-spring) is moving in a SHM. From expression (17) one can derive the formulas for circular frequency and the period of this SHM as

$$\boxed{\omega = \sqrt{\frac{k}{m}}} \qquad (18) \qquad\qquad T = \frac{2\pi}{\omega} \Rightarrow \boxed{T = 2\pi\sqrt{\frac{m}{k}}} \qquad (19)$$

Note: ***T*** depends on ***k*** and ***m***, but <u>not</u> on ***A*** (*SHM requirement*).

Energy in a horizontal spring-block SHM

Let's refer to the upper example (horizontal spring-block set). In a SHM there is no friction (*ideal model*). So, there are three forces acting on the block; the ***elastic force***, the ***weight*** and the ***normal*** force. The elastic force and weight are ***conservative forces*** and one can define a " ***system*** " block-spring-earth. By selecting Ox horizontal and Oy vertical, one can find (mechanics) that potential energy of this system is

$$\boldsymbol{U = U_{el} + U_g = \tfrac{1}{2} kx^2 + mgh} \qquad (20)$$

The normal force is the only external force acting on this system. As its work is zero, $W_{ext} = \Delta E = 0$ and we deal with an ***isolated system***. The mechanical energy of this system is $\boldsymbol{E = U_{el} + U_g + K}$ where $\boldsymbol{U_g}$ doesn't change during the oscillations. If one fix $\boldsymbol{U_g} = 0$ at support level, the <u>*total mechanical energy*</u> of

the system becomes $$\boldsymbol{E = U_{el} + K} = \tfrac{1}{2}\boldsymbol{kx^2} + \tfrac{1}{2}\boldsymbol{mv^2} \qquad (21)$$
As $\Delta E = 0$, this quantity must remain constant during oscillations.

Now, let's substitute the ***SHM*** expressions for displacement and velocity at this expression of $\boldsymbol{E}$.

$$x = Asin(\omega t + \varphi_0) \qquad U_{el} = \frac{1}{2}kx^2 \Rightarrow U_{el} = \frac{1}{2}kA^2\sin^2(\omega t + \varphi_0) \qquad (22)$$

$$x' = v = A\omega\cos(\omega t + \varphi_0) \quad K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2A^2\cos^2(\omega t + \varphi_0)$$
$$K = \frac{1}{2}m\frac{k}{m}A^2\cos^2(\omega t + \varphi_0) = \frac{1}{2}kA^2\cos^2(\omega t + \varphi_0) \qquad (23)$$

and $$\Longrightarrow E = U_{el} + K = \frac{1}{2}kA^2\sin^2(\omega t + \varphi_0) + \frac{1}{2}kA^2\cos^2(\omega t + \varphi_0)$$

$$\Longrightarrow E = \frac{1}{2}kA^2[\sin^2(\omega t + \varphi_0) + \cos^2(\omega t + \varphi_0)] \Longrightarrow \boxed{E = \frac{1}{2}kA^2} \qquad (24)$$

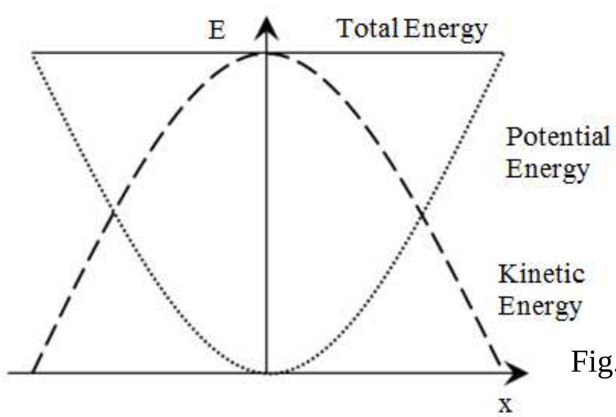


Fig.5

The graphs in fig.5 show the evolution of potential and kinetic energy with x- value. As seen from this graph, each of them has positive values and their sum, i.e. <u>**mechanical energy of system**</u> spring-block $\boldsymbol{E} = \tfrac{1}{2}\boldsymbol{kA^2}$ remains constant in time. This means that for any position " x " , $U(x) + K(x) = \tfrac{1}{2}\boldsymbol{kA^2}$
The potential energy of this system " $\boldsymbol{U} = \tfrac{1}{2}\boldsymbol{kx^2}$ " is a parabolic function of displacement $\boldsymbol{x}$ (see Fig.5). One says that SHM is characterized by a parabolic " *potential well* ".
Actually, the form of the "potential well" is defined by the force at its origin. The parabolic potential is met whenever an elastic restoring force applies on an oscillating system. The parabolic wells and elastic restoring forces are used as a first approximation for the study of oscillations. Other types of forces may act on a system and the potential wells may not be parabolic.

Simple Harmonic Oscillations(***SHO***)

In a *Simple Harmonic Motion* (***SHM***), the displacement represents a real ***shift in space***. Meanwhile, any physical parameter (electric current, temperature, pressure, etc…) may oscillate in time as a harmonic function. In this case, one says that the parameter performs a ***Simple Harmonic Oscillation*** (***SHO***). Therefore *a SHM is in fact a SHO where the displacement is a real shift in space.*

When deriving the differential equation (10), we didn't mention any special requirement for the physical nature of ***y(t)***. Actually, ***equation (10) is valid for any SHO*** and is known as the ***SHO equation***.

SHO equation $\frac{d^2y}{dt^2} = -\omega^2 * y$ or $\frac{d^2y}{dt^2} + \omega^2 * y = 0$

If the " *displacement* from *equilibrium value* " of a *physical quantity* obeys to this type of differential equation, one may affirm straight away that this parameter is performing a ***SHO***.

- There are three main types of **harmonic oscillations**:
  a) Simple Harmonic Oscillations (***SHO***); no energy loss = ***ideal model***
  b) Damped Harmonic Oscillations (***DHO***); energy loss = ***real life model***
  c) Forced Harmonic Oscillations (***FHO***); external force compensates for lost energy = ***real life model***

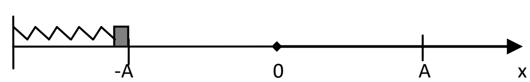
We will study DHM and FHM characteristics in the next section. Note that the results are valid for any DHO and FHO because the form of the mathematical expressions is the same.

Actually, almost all the results that we obtain from studying SHM are also valid for SHO. Also, the total energy for any SHO has the same mathematical form as the expression (24), except that instead of an elasticity constant "***k***", there is another "***restoring constant"***.

Important: In any SHO, the total energy E $\propto A^2$.

Example#1-SHM. *The motion of a 0.25kg block around its equilibrium position in a spring-block system is given by the function y = -2 sin(4t+π/2)[m].*

Note: If there is a '-' sign in front of SHM function, 'transfer its effect' to the phase constant by adding or subtracting 'π' inside brackets as sin(x +/- π) = -sin(x). As one prefers to work with a phase constant value between -π and + π, in our exercise, by using "- π", one get $y = y_1 = 2\sin(4t-\pi/2)[m]$.



a) Find the amplitude of oscillations. From the function one gets A=2[m]

b) Find the angular(or circular) frequency of oscillations. From the function one gets ω = 4[r/s]

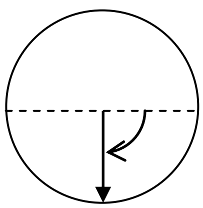
c) Find the period and natural frequency of oscillations. T=2π/ω=6.28/4=1.57[s]
f = ω/ 2π =4/6028 =0.64[Hz]

d) Find the spring constant As $\omega^2 = k / m$;
$k = \omega^2 m = 4^2 * 0.25 = 16 * 0.25 = 4[N/m]$

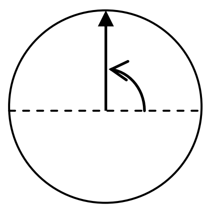
e) Find the first times (after t = 0s) when the spring has *maximum extension* and *maximum compression*.

Procedure: If there is a question about time, one must find *at first* the *corresponding phase* $\varphi(t) = \omega t + \varphi_o$ at that time. Next one uses the relation between the phase and time to find the requested time. During this process is important to start by drawing the position of phasor at t = 0s and its position at requested time.

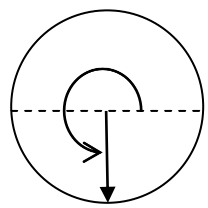
In this problem, at t=0s $\varphi = \omega * 0 - \pi/2$ which means that $\varphi_o = -\pi/2$. So, one starts by drawing the phasor at t=0s. As the problem asks for maximum extension /compression **after** t=0s ($y_{1/2} = +/- 2[m]$) , it is clear that this conditions are realised for the time (t_1) when the phasor is vertically up ($\varphi_1 = +\pi/2$) and time (t_2) when it is anew vertically down ($\varphi_2 = 3\pi/2$).



t=0s; $\phi = \varphi_o = -\pi/2$



t_1; $\phi_1 = +\pi/2$



t_2; $\phi_2 = +3\pi/2$

Next:

$\varphi_1 = +\pi/2 = 4t_1 - \pi/2$ gives $t_1 = 0.785$s

$\varphi_2 = +3\pi/2 = 4t_2 - \pi/2$ gives $t_2 = 1.57$s

e) Write the displacement as a "cos" function. As $\sin(x-\pi/2) = -\cos(x)$ and $-\cos(x) = \cos(x+\pi)$, for $x = 4t$ one get $\sin(4t-\pi/2) = -\cos 4t = \cos(4t+\pi)$. So, we may write the SHM function as $\mathbf{y = 2\cos(4t+\pi)[m]}$

f) Find block velocity $v = dy/dt = d/dt\ (-2\sin(4t+\pi/2)) = (-2)(4)\cos(4t+\pi/2) = -8\cos(4t+\pi/2)[m/s]$

i) Find maximum magnitude of spring force. $F_{el} = -ky$ and $F_{max} = abs(ky_{max}) = kA = 4*2 = 8N$

g) Find the mechanical energy of system, its maximum kinetic energy and maximum speed of block.

$E = K+U = K_{max} = U_{max}$. So, $E = U_{max} = 0.5kA^2 = 0.5*4*2^2 = 8J$; Also, $E = K_{max}$. So $K_{max} = 8J$

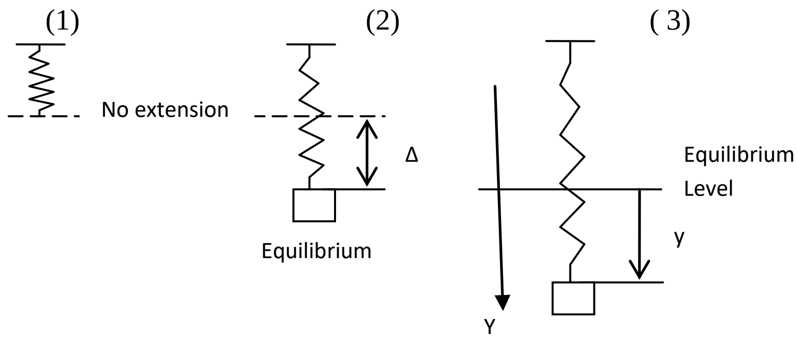
As $K_{max} = 1/2 m v_{max}^2$ it comes out that $v_{max} = [2*8/0.25]^{1/2} = 8m/s$

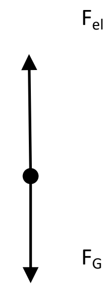
Or straight away from velocity expression at "f" one gets $speed_{max} = [abs(v)]_{max} = 8m/s$ and next, finds out K_{max} and E .

Example#2-SHM. *VERTICAL SHM FOR A BLOCK-SPRING SYSTEM*

When a block with mass m=1kg is hooked at the free end of a vertical spring it is extended by 30cm. Then, one pulls down the block and leave it free. The system spring-block starts to oscillate.

a) *Show that, if the air friction is neglected, these oscillations are a SHM*
b) *Find their period.*





a_1) At state (2) the block is at equilibrium. So, $\vec{F}_{Net} = \vec{F}_{elastic} + \vec{F}_G = 0$ (1)
We project this equation on Oy and get $\boldsymbol{-k\Delta + mg = 0}$ or $\boldsymbol{k\Delta = mg}$ (2)

a_2) At state (3) the block is moving and the second law of Newton is written

$$\vec{F}_{Net} = \vec{F}_{elastic} + \vec{F}_G = m\vec{a} \qquad (3)$$

We project on Oy and get $\boldsymbol{-k(\Delta+y) + mg = ma}$ (4)
By substituting (2) at (4) we get $\boldsymbol{(-k\Delta) - ky + mg = (-mg) - ky + mg = ma}$
and $\boldsymbol{-ky = ma}$ which can be written as $\boldsymbol{a = -(k/m)*y}$ Then, by noting $\boldsymbol{k/m = \omega^2}$
we get to equation $\frac{d^2y}{dt} = -\omega^2 y$ (5) which is the SHM equation.

b) The period of these oscillations is $T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{k}{m}}} = 2\pi\sqrt{m/k}$. One can find k-value by using Hook's law for the initial extension. From (2) $\boldsymbol{k = mg/\Delta = 1*9.8/0.3 = 32.6N/m}$
Then, one get $T = 2\pi\sqrt{1/32.7} = 1.1s$

Note: The parameters of vertical SHM oscillations of a bloc-spring system are the same as if the system was oscillating without friction on a horizontal plan.