

2.1. SIMPLE PENDULUM, ANOTHER EXAMPLE OF SHM

-The simple pendulum is a **physic's model** one uses when studying the "small angle" oscillations of an object hanged at the end of a rope. In this model, the object is presented by a **material point** with mass "**m**" tied at the end of a **mass-less** string of **length "L"**. One assigns a **positive** direction (CCW in general) for the rope **angle "θ"** to its **vertical position** and labels as "**s**" the corresponding **arch displacement** counted from the equilibrium (as shown in fig.1). Also, one neglects the air friction effect.

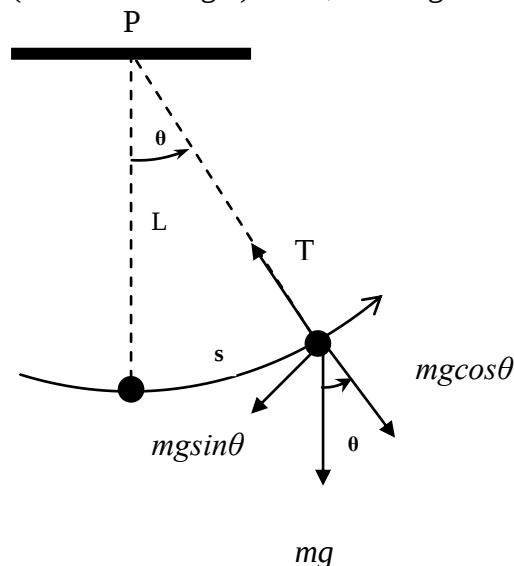


Figure 1

One writes the **second law of Newton for the rotation** of a **particle** around an axis Oz normal to page and passing by P-point

$$\vec{\tau}_{net} = I \vec{\alpha} \quad (1)$$

By projecting eq.(1) on Oz axe (**pointing out of page**) one get

$$\tau_{net-Oz} = (mL^2) * \alpha \quad (2)$$

As $\alpha = \frac{d^2\theta}{dt^2}$ (3) and only the "**restoring component**" of weight " **$mg \sin \theta$** " has "**non zero**" **torque** $\tau_{net-Oz} = -(mg \sin \theta) * L$

the relation (2) becomes $-mg \sin \theta * L = mL^2 * \frac{d^2\theta}{dt^2}$ (4)

Next, after cancelling "**m**" and "**L**" $-g \sin \theta = L * \frac{d^2\theta}{dt^2}$ (5)

For small angles, $\sin \theta \sim \theta$ and the relation (5) transforms to $-g\theta = L \frac{d^2\theta}{dt^2}$ or $\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta$ (6)

This equation fits exactly to **equation of a SHM** if one assigns $\theta = x$ and $\frac{g}{L} = \omega^2$ (7)

Then the **phasor** attached to the oscillation of the "**displacement = angle θ**" rotates at a **circular frequency**

$\omega = \sqrt{\frac{g}{L}}$. Also, from the results of SHM modelling, one derives that:

- The pendulum angle "**θ**" oscillates as a harmonic function of time given by expression

$$\theta = \theta_{max} \sin(\omega t + \varphi_0) \quad (8)$$

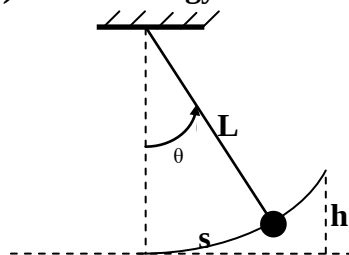
- The period of these oscillations is $T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{L}{g}}$ (9)

Notes: a) Equation (6) tells that the "**displacement θ**" oscillates as a SHM with a period "**T**" given by expression (9).

b) Do not mix the **pendulum angle θ** with the **phase angles** ($\Phi(t) = \omega t + \varphi_0$, φ_0)

c) The **translational velocity** of mass "**m**" is $v = L * d\theta/dt$ because $s = L * \theta$ (see fig.1).

Example-Pendulum Given that the angle to vertical of a simple pendulum with mass **0.5kg** changes in time as $\theta(t) = 0.5[\text{rad}]\cos(0.5\pi * t + \frac{\pi}{2})$ find: **a)** The **period** of oscillations; **b)** The **length** of pendulum; **c)** the **maximum angle** reached in degree; **d)** the **position** and **linear velocity** of bob at $t = 0$ **e)** **kinetic energy** of bob when the angle is $\theta = 0.2\text{rads}$; **f)** **total mechanic energy** of pendulum



a) $\omega = 0.5\pi$. So, $T = 2\pi/\omega = 2\pi/0.5\pi = 4\text{sec}$.

b) $\omega^2 = g/L$. So, $L = g/\omega^2 = 9.8/(0.5*3.14)^2 = 3.97[\text{m}]$

c) From function $\theta_{\max} = 0.5[\text{r}] = 0.5[\text{r}] * (180^\circ/3.14[\text{r}]) = 28.66^\circ$

d) At $t = 0\text{s}$ $\theta(0) = 0.5\cos(0+\pi/2) = 0.5\cos(\pi/2) = 0[\text{r}]$.

The linear(or translational) velocity is calculated from the displacement. So, one should refer to the translational displacement "s" from lowest level (where $s=0$) and counted as "+" to the right side. As $s = L*\theta$, one get

$v(t) = ds/dt = d(L*\theta) / dt = Ld\theta/dt = 3.97* 0.5*(-0.5\pi)\sin(0.5\pi t + \pi/2) = -3.12\sin(0.5\pi t + \pi/2) \equiv -3.12\sin\phi(t)$
Then, at $t=0$ one gets $v(0) = -3.12\sin(\pi/2) = -3.12[\text{m/s}]$. As $v < 0$ the bob starts its motion left side.

e) $K(\theta = 0.2\text{r}) = m*v^2(\theta=0.2\text{r})/2 = 0.25* v^2(\theta=0.2\text{r})$. So, one has to find v-value when $\theta = 0.2\text{r}$. One starts by finding the phase $\phi(t)$ that corresponds to $\theta = 0.2\text{r}$; then, one calculates $v = -3.12\sin\phi(t)$ for this value of phase. Starting by $0.2 = 0.5\cos\phi$ or $\cos\phi = 0.2/0.5 = 0.4$, one gets $\phi = \arccos 0.4 = +/- 1.16[\text{r}]$. Next, one uses this phase value at $v = -3.12\sin\phi(t)$ and gets $K(\theta=0.2\text{r}) = 0.25*v^2(\theta=0.2\text{r}) = 0.25*[-3.12\sin(+/-1.16)]^2 = 2.04 \text{ Joules}$

f) Two ways **f.1)** $E = K_{\max} = K(\theta=0\text{r}) = (0.5/2)*v_{\max}^2 = 0.25*(3.12)^2 = 2.4 \text{ Joules}$

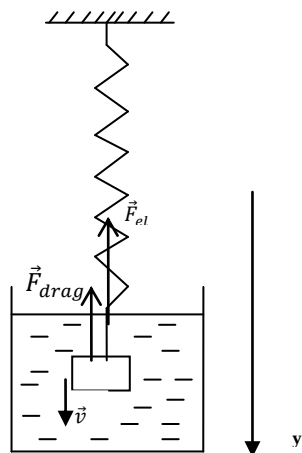
or **f.2)** $E = U_{\max} = U(\theta_{\max}) = mgh_{\max} = mgL(1 - \cos\theta_{\max}) = 0.5*9.8*3.97(1 - \cos 0.5\text{r}) = 2.4\text{J}$.

Note that $h_{\max} = L - L\cos\theta_{\max} = L(1 - \cos\theta_{\max})$

2.2 DAMPED OSCILLATIONS

-In SHO and SHM models (*S stands for simple*) there is **no energy loss** with time and the oscillations "continue to infinity". But, in a real system, due to the **friction** with surrounding medium, after a certain time, oscillations will stop. This **damping effect** appears as a *decrease of oscillations energy* and (as $A \sim E^{1/2}$) *amplitude* with time.

-One may model the *damping effects* by referring to oscillations of a spring-block system when the block is moving inside a liquid(fig.4). One knows that, in this case, the liquid exerts a *drag force* on the block. The *drag force* is directed *opposite to direction* of block motion (i.e. opposite to velocity) and, for **moderate speed**, its magnitude is proportional to magnitude of velocity. So, one gets



$$\vec{F}_{drag} = -b * \vec{v} = -b \frac{dy}{dt} \hat{j} \quad (10)$$

$b[\text{Kg/s}]$ is the **damping constant** of liquid on block oscillations

$v[\text{m/s}] = dy/dt$ is the block **velocity**, $\hat{j}$ -unit vector along Oy axis.

$$\text{Then, } \vec{F}_{Net} = \vec{F}_{el} + \vec{F}_{drag} = -ky\hat{j} - b \frac{dy}{dt} \hat{j} = \left(-ky - b \frac{dy}{dt}\right) \hat{j}$$

($\vec{F}_g$ action is canceled by $\vec{F}_{el=0}$ of spring extension at equilibrium - see ex.#2 at lecture#1)

and the second law of Newton $\vec{F}_{NET} = m\vec{a}$ projected on

$$\text{Oy axis takes the form } -ky - b \frac{dy}{dt} = m \frac{d^2y}{dt^2} \quad (11)$$

Fig 4

One can see that the **equation** for the **SHM** of **block-spring** system

$$m \frac{d^2 y}{dt^2} = -ky \quad \text{transforms to} \quad m \frac{d^2 y}{dt^2} = -ky - b \frac{dy}{dt} \quad (12)$$

in presence of damping. In general, one rewrites (12) in form

$$m \frac{d^2 y}{dt^2} + ky + b \frac{dy}{dt} = 0 \quad (13)$$

The mathematical equation of type (13) is valid for all damped harmonic oscillations "**DHO**" (mechanic, electric...) and it is very well studied. *Its oscillating solution is an harmonic function which amplitude decreases exponentially with time (see its graph in fig.5).*

Essentially, the "**displacement**" function for a DHM has the form:

$$y(t) = A'(t) \sin(\omega' t + \varphi) \quad (14) \quad \text{where the } \textbf{damped amplitude} \text{ is} \quad A'(t) = A_0 e^{-\lambda t} \quad (15)$$

$$\text{By substituting (14) and (15) in (13) one finds out the "decay constant } \lambda" \quad \lambda = b/2m \quad (16)$$

$$\text{and the } \textbf{damped circular frequency } \omega' \quad \omega' = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2} \quad (17)$$

Note: DHO circular frequency ω' is smaller than SHO circular frequency; $\omega' < \omega_0 = \sqrt{\frac{k}{m}}$ and $T' > T_0$.

Actually, the solutions of equation (13) correspond to three types of different motions:

-Under-damped oscillations that happen if the damped circular frequency ω' at (17) is **positive**, i.e. if

$$\omega' > 0 \rightarrow \omega_0^2 - \left(\frac{b}{2m}\right)^2 > 0 \rightarrow \omega_0 > \left(\frac{b}{2m}\right) \rightarrow b < 2m\omega_0 \quad (18)$$

This is the case of oscillations that are lost with the time due to a *small damping effect*.

**Under-damped
Oscillations**

$$A'(t) = A_0 e^{-(b/2m)t}$$

$$T' = 2\pi / \omega'$$

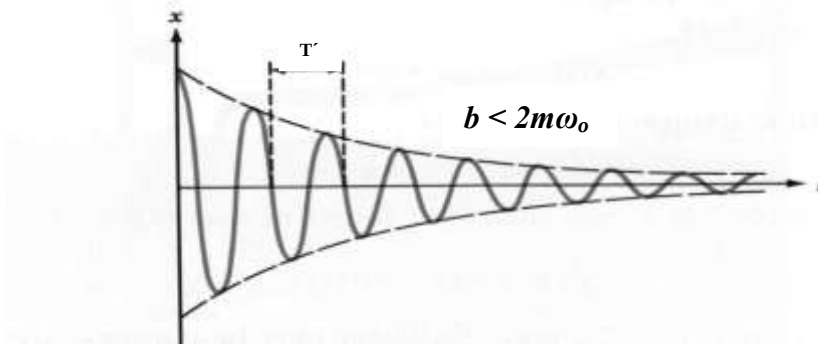


Fig 5 The evolution of displacement with time in an under damped oscillation.

Critically-damped motion (no oscillations) happens when the angular frequency $\omega' = 0$;

$$\omega' = 0 \rightarrow \omega_0^2 - \left(\frac{b}{2m}\right)^2 = 0 \rightarrow \omega_0 = \left(\frac{b}{2m}\right) \rightarrow b = 2m\omega_0 \quad (19)$$

Critical damping produces a **return to equilibrium** motion at the **shortest time** (fig.6).

(Ex: electrical device needle). If $b \lesssim 2m\omega_0$ the system is "less than critical" but **not really under-damped**. So, it performs a few oscillations before stopping (ex. cars' suspension).

Over-damped motion (no oscillations) if the angular frequency ω' is an **imaginary number**

i.e. when
$$\omega' = im \rightarrow \omega_0^2 - \left(\frac{b}{2m}\right)^2 < 0 \rightarrow \omega_0 < \left(\frac{b}{2m}\right) \rightarrow b > 2m\omega_0 \quad (20)$$

In this case the system **returns to equilibrium slowly**.(ex. Heavy doors that close slowly)

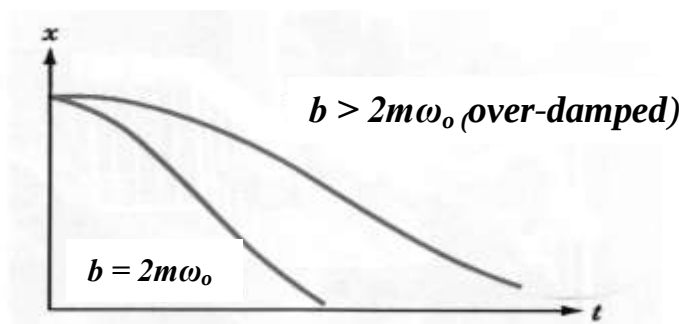


Figure 6

Example-DHM. For a given under damped oscillator with $m=0.5\text{kg}$, $k=8.5\text{N/m}$, $b=0.4\text{kg/s}$, find:

- The period of oscillations.
- How long does it take for the amplitude to drop to half of its initial value.
- How long does it take for the mechanical energy to drop to one half of its initial value.
- What is the ratio (A_5/A_0) of the amplitude after 5 cycles to its initial value.

a) $\omega_0^2 = k/m = 8.5/0.5 = 17[\text{r/s}]^2$; $\omega_0\sqrt{17} = 4.123\text{r/s}$ and $T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{4.123} = 1.524\text{s}$
 As $\omega' = [\omega_0^2 - (b/2m)^2]^{1/2}$ and $b/2m = 0.4/2*0.5 = 0.4 (\equiv \lambda)$
 $\omega' = [17 - 0.4^2]^{1/2} = 4.104 \text{ r/s}$ and $T' = 2\pi/\omega' = 1.531\text{s} (>T_0)$

b) $A = A_0/2 = A_0 e^{-(\frac{b}{2m})T_{1/2}}$ $0.5 = e^{-0.4*T_{1/2}}$

$e^{0.4*T_{1/2}} = 2$ and $0.4 * T_{1/2} = \ln 2 = 0.693$

$T_{1/2} = 0.693/0.4 = 1.73[\text{s}]$. So, after 1.73s the amplitude is twice smaller.

c) At $t_1 \rightarrow E_1 = (0.5*kA_1^2) = 0.5E_0 = 0.5(0.5*kA_0^2) = 0.25k A_0^2$. So, $A_1^2 = 0.5A_0^2$ and $A_1 = 0.707A_0$ or $A_1 = A_0 e^{-0.4*t_1} = 0.707A_0$. So, $e^{0.4*t_1} = 1 / 0.707 = 1.414$ and $0.4*t_1 = \ln 1.414 = 0.346$ and $t_1 = 0.87\text{s}$

d) $t = 5*1.531\text{s} = 7.655\text{s}$. So, $A(7.655\text{s}) = A_0 e^{-0.4*7.655}$ and $A(7.655\text{s}) / A_0 = e^{-0.4*7.655} = 0.0468$

2.3 FORCED OSCILLATIONS

- A mechanical oscillator has its characteristic **natural circular frequency** ($\omega_0 = \sqrt{k/m}$) which corresponds to its *free oscillations* (ideal model). In reality, there is always *damping* due to the different interactions and the system loses energy. If $b < 2m\omega_0$ (**under damped** situation), it achieves a **DHM** at **circular frequency**

$\omega' = \sqrt{\omega_0^2 - (b/2m)^2}$ and the oscillations *disappear* with time. If $b \geq 2m\omega_0$ there is an **over or critically damped** situation; there is **no oscillation**.

- One can make oscillations continue by compensating the energy loss in a *periodic* way. To keep a **steady-state oscillations** one must apply an external¹ **periodic force**. Note that in this case one is dealing with a **FHO**, *forced* (or a *driven*) oscillation; not a **SHM** or **DHM**.

Assuming that the external periodic force is

$$\vec{F}_{driv} = F_0 \cos(\omega_{dr} * t) \hat{j} \quad (21)$$

the 2nd law for a *driven oscillation* undergoing damping is

$$\vec{F}_{el} + \vec{F}_{res} + \vec{F}_{driv} = m \vec{a} \quad (22)$$

By projecting equation (22) on an axis parallel to the direction of motion (see fig.4)

one gets the expression $m \frac{d^2 y}{dt^2} = -ky - b \frac{dy}{dt} + F_0 \cos(\omega_{dr} * t)$ (23)

which can be transformed into $m \frac{d^2 y}{dt^2} + ky + b \frac{dy}{dt} = F_0 \cos(\omega_{dr} * t)$ (24)

The equation (24) is known as the **equation of driven oscillations**. The solution of equation (24) is not object of this course but the main results that come out of solution are as follows:

A *driven oscillator* performs an harmonic motion with two main characteristics;

- its **circular frequency** is equal to that of the *external driving force* ($\omega = \omega_{dr}$).
- its **amplitude** is *maximum* if $\omega_{dr} \cong \omega'$ and it depends on the *damping constant* " b ".

A given oscillator has a given set of values (m, k, ω_0, b). While ω_0 -value depend only on oscillator, the b -value depends on the surrounding mediums, too. The graphs in fig.7 (known as **resonance curves**) show the **evolution** of the **amplitude of a driven oscillator** vs. the **ratio** (ω_{dr}/ω_0) of *driving frequency* for different *damping* situations. All these curves present a maximum that happens when the **driving frequency** (ω_{dr}) is close to the **natural circular frequency** (ω_0) of oscillator. One says that a **resonance** is produced in a system when the **amplitude** of oscillations gets the **maximum** value on the graph $A=A(\omega)$. The **resonance** of the same driven oscillator (same $k, m, \omega_0, F_0, \omega_{dr}$) is more **pronounced** (i.e. **larger amplitude**) for **low damping** (b - small) and may even **disappear** for **high damping** (b - very large).

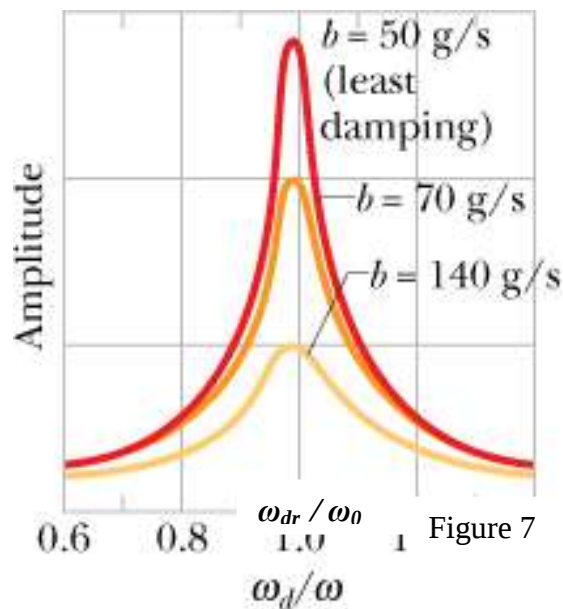


Figure 7

¹ To the oscillating system