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2.1. SIMPLE PENDULUM, ANOTHER EXAMPLE OF SHM

-The simple pendulum is a ***physic's model*** used for the study of "*small angle*" oscillations of an object tied at the end of a rope. One models the object as a *material point* with *mass "**m**"* fixed at the end of a *mass-less* string with *length "**L**"*. Next, one assigns a *positive* direction (CCW in general) for the rope *angle* " ***θ*** " *to its vertical position* and the corresponding *displacement on arch* "**s**" counted from equilibrium(as shown in fig.1). Also, one neglects the air friction effect.

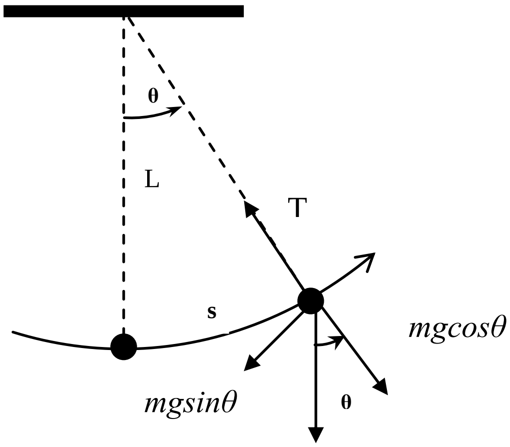


Figure 1

By applying the *second law of Newton for the rotation* of *particle* around an axis Oz normal to page and passing by P-point, one get

$$\vec{\tau}_{net} = I\vec{\alpha} \qquad (1)$$

By projecting eq.(1) on Oz axe (pointing out of page) one get

$$\tau_{net-Oz} = mL^2\alpha \qquad (2)$$

As $\alpha = \frac{d^2\theta}{dt^2}$ (3) and only the " ***restoring** component* " of weight "***mgsinθ***" has "non zero" ***torque*** $\tau_{net-Oz} = -(mg\sin\theta)*L$

the relation (2) becomes $-mg\sin\theta * L = mL^2 * \frac{d^2\theta}{dt^2}$ (4)

Next, after cancelling "*m*" and "*L*" $-g\sin\theta = L*\frac{d^2\theta}{dt^2}$ (5)

For small angles, sin θ ~ θ and relation (5) transforms to $-g\theta = L\frac{d^2\theta}{dt^2}$ or $\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta$ (6)

This equation fits exactly to ***equation of a SHM*** if one assigns ***θ = x*** and $\frac{g}{L} = \omega^2$ (7)

Then the ***phasor*** attached to the oscillation of the "***displacement = angle θ***" rotates at a ***circular frequency*** $\omega = \sqrt{\frac{g}{L}}$. Also, from the results of SHM modelling, one derives that:

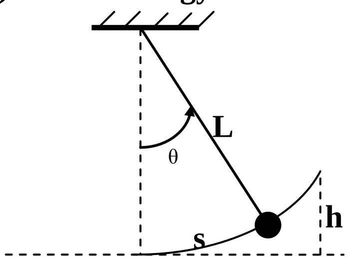
- The pendulum angle *"θ"* oscillates as a harmonic function of time given by expression

$$\theta = \theta_{max}\sin(\omega t + \varphi_0) \qquad (8)$$

- The period of these oscillations is $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$ (9)

Notes: ***a)*** Equation (6) tells that the "***displacement θ***" oscillates as a SHM with a period " *T* " given by expression (9).
b) Do not mix the ***pendulum angle θ*** with the ***phase angles*** ($\Phi(t) = \omega t + \varphi_0$, φ_0)
c) *The translational **velocity** of mass "**m**" is **υ = L* dθ/dt** because **s = L*θ** (see fig.1).*

Example-Pendulum Given that the angle to vertical of a simple pendulum with mass **0.5kg** changes in time as $\theta(t) = 0.5[rad]\cos\,(0.5\pi * t + \frac{\pi}{2})$ find: ***a)*** The **period** of oscillations; ***b)*** The **length** of pendulum; ***c)*** the **maximum angle** reached in degree; ***d)*** the **position** and **linear velocity** of bob at t = 0 ***e)*** **kinetic energy** of bob when the angle is **θ = 0.2***rads* ; ***f)*** **total mechanic energy** of pendulum



a) ω = 0.5π . So, T = 2π/ω = 2π/0.5π = 4sec.
b) $\omega^2 = g/L$. So, $L = g/\omega^2 = 9.8/(0.5*3.14)^2 = 3.97$[m]
c) From function θ_{max}= 0.5[r]= 0.5[r] *(180°/3.14[r])= 28.66°
d) At t= 0s θ(0) = 0.5cos(0+π/2) = 0.5cos(π/2)= 0[r] .
The linear(or translational) velocity is calculated from the displacement. So, one should refer to the translational displacement "s" from lowest level (where s=0) and counted as "+" to the right side. As s = L*θ , one get

v(t) = ds/dt = d(L*θ) / dt = Ldθ/dt =3.97* 0.5*(-0.5π)sin(0.5πt+π/2) = -3.12sin(0.5πt+π/2)≡ -3.12sinφ(t) (*)
Then, at t=0 one gets v(0) = -3.12sin(π/2)= -3.12[m/s]. As υ < 0 the bob starts its motion left side.

e) $K(\theta = 0.2r) = m*v^2(\theta=0.2r)/2 = 0.25* v^2(\theta=0.2r)$. So, one has to find v-value when θ = 0.2r.
One starts by finding the phase φ(t) that corresponds to θ = 0.2r ; then, one calculates the expression (*) for this value of phase. Starting by 0.2 = 0.5cosφ or cosφ = 0.2/0.5 = 0.4, one gets φ = arscos0.4 = +/- 1.16[r]
Next, one uses this phase at (*) and gets $K(\theta=0.2r) = 0.25*[-3.12\sin(+/-1.16)]^2$= 2.04 Joules

f) Two **ways** **f.1)** $E = K_{max} = K(\theta=0r) = (0.5/2)*v^2_{max} = 0.25 *(3.12)^2$ = 2.4 Joules
or **f.2)** $E = U_{max} = U(\theta_{max}) = mgh_{max} = mgL(1-\cos\theta_{max}) = 0.5*9.8*3.97(1-\cos 0.5r)$ = 2.4J.
Note that $h_{max} = L - L\cos\theta_{max} = L(1-\cos\theta_{max})$

2.2 DAMPED OSCILLATIONS

-In SHO and SHM models (*S stands for simple*) there is ***no energy loss*** with time and oscillations "continue to infinity". But, in a real system, due to the **friction** with surrounding medium, after a certain time, oscillations will stop. This *damping effect* appears as a *decrease of oscillations energy* and (as ***$A \sim E^{1/2}$)*** *amplitude* with time.

-One may model the *damping effects* by referring to oscillations of a spring-block system when the block is moving inside a liquid(fig.4). One knows that, in this case, the liquid exerts a *drag force* on the block. The *drag force* is directed *opposite to direction* of block motion (i.e. opposite to velocity) and, for ***moderate speed,*** its magnitude is proportional to magnitude of velocity**.** So, one gets

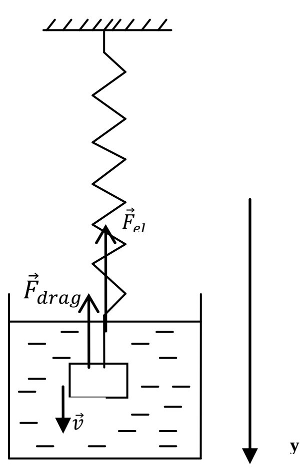


Fig 4

$$\vec{F}_{drag} = -b * \vec{v} = -b\frac{dy}{dt}\hat{j} \qquad (10)$$

b[Kg/s] is the ***damping constant*** of *liquid on block* oscillations
υ[m/s] = ***dy/dt*** is the block ***velocity***, $\hat{j}$-unit vector along Oy axis.

Then, $\vec{F}_{Net} = \vec{F}_{el} + \vec{F}_{drag} = -ky\hat{j} - b\frac{dy}{dt}\hat{j} = \left(-ky - b\frac{dy}{dt}\right)\hat{j}$
(*$\vec{F}_g$ action is canceled by $\vec{F}_{el-0}$ of spring extension at equilibrium - see ex.#2 at lecture#1)*)
and the second law of Newton $\vec{F}_{NET} = m\vec{a}$ projected on Oy axis takes the form

$$-ky - b\frac{dy}{dt} = m\frac{d^2y}{dt^2} \qquad (11)$$

One can see that the **equation** for the **SHM** of **block-spring** system

$$m\frac{d^2y}{dt^2} = -ky \qquad \textbf{transforms to} \qquad m\frac{d^2y}{dt^2} = -ky - b\frac{dy}{dt} \tag{12}$$

in presence of damping. In general, one rewrites (12) in form

$$m\frac{d^2y}{dt^2} + ky + b\frac{dy}{dt} = 0 \tag{13}$$

The mathematical equation of type (13) is valid for all damped harmonic oscillations "***DHO***" (mechanic, electric..,) and it is very well studied. *Its oscillating solution is an harmonic function which amplitude decreases exponentially with time(see its graph in fig.5).*

Essentially, the " ***displacement*** " function for a DHM has the form:

$y(t) = A'(t)\sin(\omega' t + \varphi)$ (14) where the ***damped amplitude*** is $A'(t) = A_0 e^{-\lambda t}$ (15)

By substituting (14) and (15) in (13) one gets out that ***decay constant λ*** is **$\lambda = b/2m$** (16)

and the ***damped circular frequency* ω'** is $\omega' = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$ (17)

Note: *DHO* ***circular frequency*** *ω' is* ***<u>smaller</u>*** *than SHO* ***circular frequency;*** $\omega' < \omega_0 = \sqrt{\frac{k}{m}}$ *and* ***T' > To***.

Actually, the solutions of equation (13) correspond to three types of different motions:

-***<u>Under-damped</u>* oscillations** that happen if the *<u>damped circular frequency</u>* **ω´** at (17) is ***positive***, i.e. if $\omega' > 0 \rightarrow \omega_0^2 - \left(\frac{b}{2m}\right)^2 > 0 \rightarrow \omega_0 > \left(\frac{b}{2m}\right) \rightarrow b < 2m\omega_0$ (18)

This is the case of oscillations that are lost with the time due to a *small damping effect* .

Under-damped Oscillations ***$A'(t) = A_o e^{-(b/2m)t}$*** ***$T' = 2\pi/\omega'$***

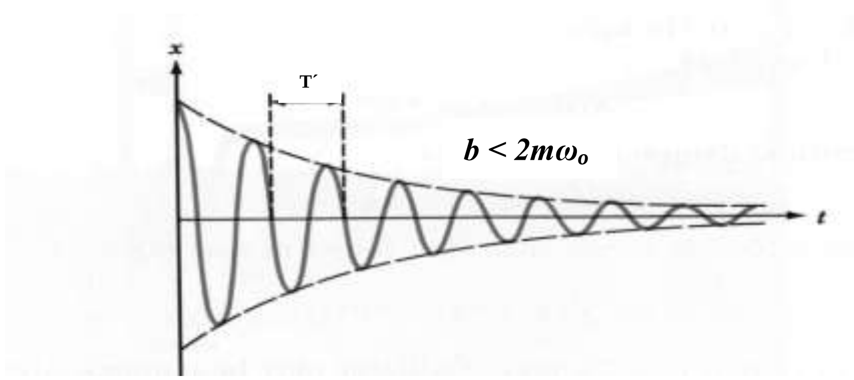


Fig 5 The evolution of displacement with time in an under damped oscillation.

Critically-damped ***motion*** ***(****no oscillations****)*** happens when the angular frequency ω' = 0;

$$\omega' = 0 \rightarrow \omega_0^2 - \left(\frac{b}{2m}\right)^2 = 0 \rightarrow \omega_0 = \left(\frac{b}{2m}\right) \rightarrow b = 2m\omega_0 \qquad (19)$$

Critical damping produces a ***return to equilibrium*** motion at the ***shortest time*** *(fig.6)*.
(Ex: electrical device needle). If $\boldsymbol{b \lesssim 2m\omega_0}$ the system is "*less than critical*" but *not really under-damped*. So, it performs a few oscillations before stopping (ex. cars' suspension).

Over-damped ***motion*** *(no oscillations)* if the angular frequency ω' is an *imaginary number*

i.e. when

$$\omega' = im \rightarrow \omega_0^2 - \left(\frac{b}{2m}\right)^2 < 0 \rightarrow \omega_0 < \left(\frac{b}{2m}\right) \rightarrow b > 2m\omega_0 \qquad (20)$$

In this case the system ***returns to equilibrium slowly***.(ex. Heavy doors that close slowly)

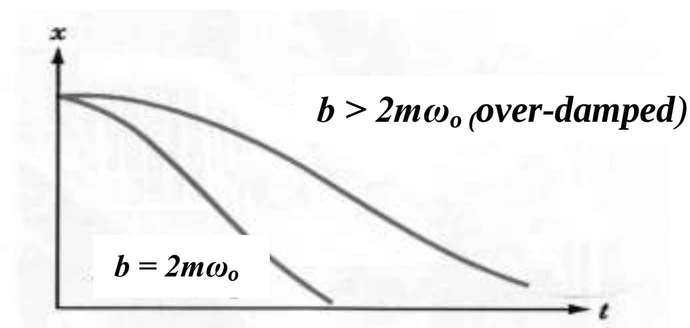


Figure 6

Example-DHM. For a given under damped oscillator with m=0.5kg, k=8.5N/m, b= 0.4kg/s, find:
a) The period of oscillations. b) How long does it take for the amplitude to drop to half of its initial value.
c) How long does it take for the mechanical energy to drop to one half of its initial value.
d) What is the ratio (A_5/A_0) of the amplitude after 5 cycles to its initial value.

a) $\omega_o^2 = k/m = 8.5/0.5 = 17[r/s]^2$; $\omega_0\sqrt{17} = 4.123r/s$ and $T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{4.123} = 1.524s$
As $\omega' = [\omega_o^2 - (b/2m)^2]^{1/2}$ and $b/2m = 0.4/2*0.5 = 0.4$ (≡ λ)
$\omega' = [17 - 0.4^2]^{1/2} = 4.104$ r/s and $T' = 2\pi/\omega' = 1.531s$ ($>T_0$)

b) $A = {A_0}/{2} = A_0 e^{-(\frac{b}{2m})T_{1/2}}$ $\qquad 0.5 = e^{-0.4*T_{1/2}}$

$e^{0.4*T_{1/2}} = 2 \quad and \quad 0.4 * T_{1/2} = \ln 2 = 0.693$

$T_{1/2}$= 0.693/0.4 = 1.73[s] . So, after 1.73s the amplitude is twice smaller.

c) At $t_1 \longrightarrow E_1 = (0.5*kA_1^2) = 0.5E_o = 0.5(0.5*kA_o^2) = 0.25k\,A_o^2$. So, $A_1^2 = 0.5A_o^2$ and $A_1 = 0.707A_o$ or $A_1 = A_o e^{-0.4*t_1} = 0.707A_o$. So, $e^{0.4*t_1} = 1 / 0.707 = 1.414$ and $0.4*t_1 = \ln 1.414 = 0.346$ and $t_1 = 0.87s$

d) t = 5*1.531s =7.655s. So, $A(7.655s) = A_o e^{-0.4*7.655}$ and $A(7.655s) / A_o = e^{-0.4*7.655} = 0.0468$

2.3 FORCED OSCILLATIONS

- A mechanical oscillator has its characteristic **natural circular frequency** ($\omega_0 = \sqrt{k/m}$) which corresponds to its *free oscillations* (ideal model). In reality, there is always *damping* due to the different interactions and the system looses energy. If ***b < 2mω₀*** (***under damped*** situation), it achieves a ***DHM*** at ***circular frequency*** $\omega' = \sqrt{\omega_0^2 - (b/2m)^2}$ and the oscillations *disappear* with time. If $b \geq 2m\omega_0$ there is an ***over*** *or* ***critically damped*** situation; there is ***no oscillation.***

- One can make oscillations continue by compensating the energy loss *in a periodic way*. To keep a ***steady-state oscillations*** one must apply an external[1] ***periodic force.*** Note that in this case one is dealing with a ***FHO***, *forced* (or a *driven)* oscillation; not a ***SHM*** or ***DHM***.

Assuming that the external periodic force is $\vec{F}_{driv} = F_0 \cos(\omega_{dr} * t)\,\hat{j}$ (21)

the 2nd law for a *driven oscillation* undergoing damping is $\vec{F}_{el} + \vec{F}_{res} + \vec{F}_{driv} = m\vec{a}$ (22)

By projecting equation (22) on an axis parallel to the direction of motion (see fig.4)

one gets the expression $m\frac{d^2y}{dt^2} = -ky - b\frac{dy}{dt} + F_0\cos(\omega_{dr} * t)$ (23)

which can be transformed into $m\frac{d^2y}{dt^2} + ky + b\frac{dy}{dt} = F_0\cos(\omega_{dr} * t)$ (24)

The equation (24) is known as the ***equation of driven oscillations***. The solution of equation (24) is not object of this course but the main results that come out of solution are as follows:

A *driven oscillator* performs an harmonic motion with two main characteristics;

a) *its circular frequency* is equal to that of the *external driving* force ($\omega = \omega_{dr}$).
b) *its* ***amplitude*** is *maximum if* $\omega_{dr} \cong \omega'$ and it depends on the *damping constant* "***b***".

A given oscillator has a given set of values (***m,k,ω₀, b***). While ω_0 -value depend only on oscillator, the ***b***-value depends on the surrounding mediums, too. The graphs in fig.7 (**known as resonance curves**) show the ***evolution*** of the ***amplitude of a driven oscillator*** vs. **the ratio** (ω_{dr}/ω_0) of *driving frequency* for different *damping* situations. All these curves present a maximum that happens when the **driving frequency (ω_{dr}) is close to the natural circular frequency (ω_0) of oscillator**. One says that *a* ***resonance*** is produced in a system when the ***amplitude*** of oscillations gets the ***maximum*** value on the graph A=A(ω). The ***resonance*** of the same driven oscillator (*same* ***k,m, ω₀ , F₀, ωdr***) is more ***pronounced*** *(i.e. larger* ***amplitude****)* for ***low damping (b - small)*** and may even ***disappear*** for ***high damping (b – very large)***.

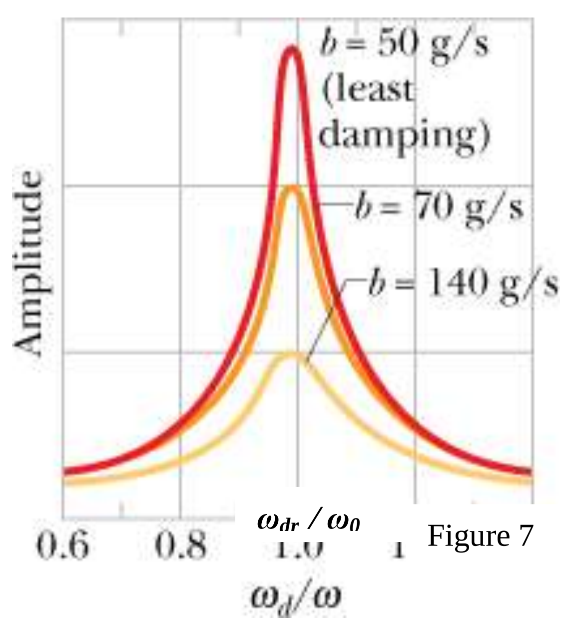


Figure 7

[1] To the oscillating system