K.Angoni, Physics Dept., Vanier College

TRIGONOMETRY BASICS

ANGLE AND ARCH UNITS

1. An ***angle*** is $\mathbf{1^0}$ if its sides **cut** an **arch** with **length equal** to **"circumference / 360" on a circle (**no matter what is its radius) ***centered at the angle crest***. The ***arch*** corresponding to the ***angle*** $\mathbf{1^0}$ *is known as the* ***arch*** $\mathbf{1^0}$.
2. An ***angle*** *is* ***1 radian*** *if its sides* ***cut an arch with length equal to the radius*** *of considered circle (*no matter what is the radius). The arch corresponding to the angle ***1 radian*** *is known as the* ***arch 1 radian.***

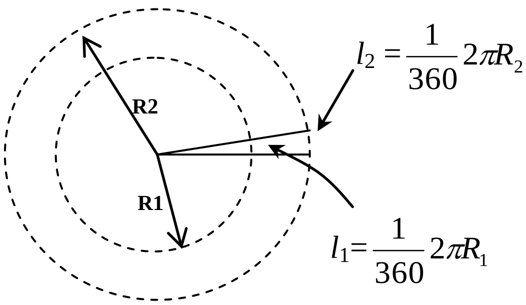


Fig.1 The angle 1^0

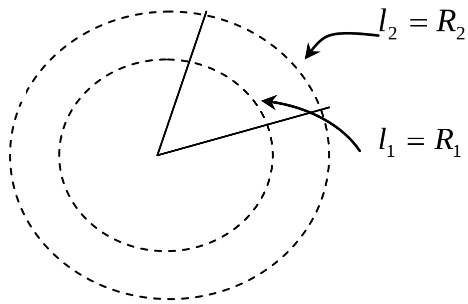


Fig.2 The angle 1 radian

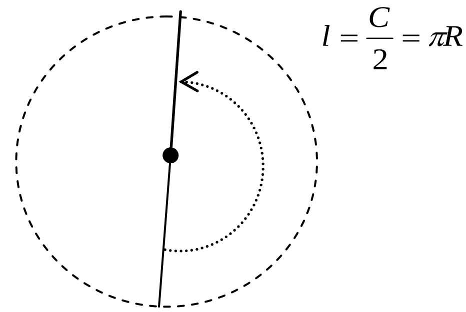


Fig.3 The angle 3.14 radians ($l/R = \pi$)

If one knows the length "***l***" of an arch on a circle with radius "***R***", one can calculate *the angle in* ***radians***, *by using the relation*

$$\alpha[radians] = \frac{l}{R} \quad \textbf{(1)}$$

Note: ***l*** *and* ***R*** *must be expressed in the same units.*

3. To convert an angle *from degrees* into *radians* or vice versa one may use the relation $\frac{\alpha[^0]}{180} = \frac{\alpha[rad]}{\pi}$ ***(2)***

Ex: 1^0 = ? radians… ⟶ $\frac{1^0}{180^0} = \frac{\alpha[r]}{\pi} \to\to \alpha = \frac{\pi[r]*1^0}{180^0} \approx 0.017453[r]$. So, $1^0 \approx 0.017453\_radians$

1radian = ? degrees... ⟶ $\frac{\alpha}{180^0} = \frac{1[r]}{\pi[r]} \to\to \alpha[^0] = \frac{180[^0]}{\pi} \approx 57^0.2958 \approx 57^0 17' 44''8$ So, $1r \approx 57.296^0$

TRIGONOMETRIC FUNCTIONS

One has introduced the ***trigonometric functions as*** *ratios of the sides lengths in a right triangle.*
But, the ***trigonometric function*** is essentially ***function of an angle***.
One places the origin of Oxy frame at the angle's crest(angle is counted CCW from Ox axis, see fig 4). The ***trigonometric functions*** are ***defined*** as ***algebraic ratios*** of the sides of an angle (***x***, ***y*** with **sign**, ***r*** *is* ***positive***) :

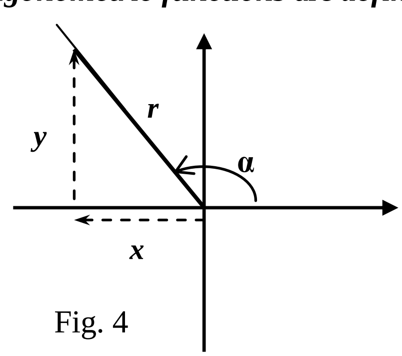


Fig. 4

$sin\boldsymbol{\alpha} = \frac{y}{r}$; $cos\boldsymbol{\alpha} = \frac{x}{r}$; $tan\boldsymbol{\alpha} = \frac{y}{x}$; (basic functions)

$cosec\boldsymbol{\alpha} = \frac{1}{sin\alpha}$; $sec\boldsymbol{\alpha} = \frac{1}{cos\alpha}$; $cotan\boldsymbol{\alpha} = \frac{1}{tan\alpha}$ (derived functions)

Note*: These functions* ***depend*** *only on the* ***angle****. For a* ***given angle α***, *if one side(say r) changes, the others(x, y) change in such a way that the values of trigonometric functions remain the same.* For simplicity, one refers to ***r = 1*** and uses the ***trigonometric circle*** (circle with radius unity) to get informed about some ***basic features*** of trigonometric functions and their evolution for different α-values.

From the ***trigonometric circle*** (***r = OM = OA=1***) in figure 5, one may see that

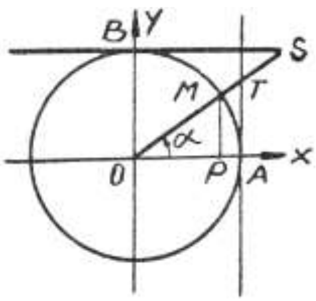


Fig. 5

sinα = PM /1= PM, cosα = OP /1= OP, tanα = AT/1= AT, and

- for	$0 < \alpha < \pi/2$	$\sin\alpha > 0$,	$\cos\alpha > 0$,	$\tan\alpha > 0$.
- for	$\pi/2 < \alpha < \pi$	$\sin\alpha > 0$,	$\cos\alpha < 0$,	$\tan\alpha < 0$.
- for	$\pi < \alpha < 3\pi/2$	$\sin\alpha < 0$,	$\cos\alpha < 0$,	$\tan\alpha > 0$.
- for	$3\pi/2 < \alpha < 2\pi$	$\sin\alpha < 0$,	$\cos\alpha > 0$,	$\tan\alpha < 0$.

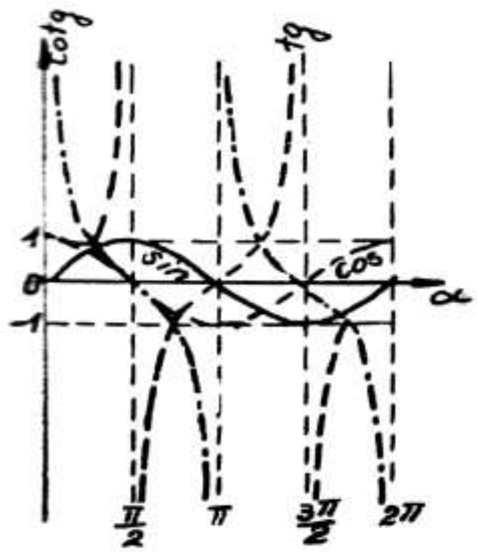


Fig. 6

Based on the trigonometric circle in fig. 5 one may find the relations
$\sin(\alpha + \pi/2) = \cos\alpha$; $\sin(\alpha - \pi/2) = -\cos\alpha$; *$\cos(\alpha + \pi/2) = -\sin\alpha$*
$\sin(\pi - \alpha) = \sin\alpha$; $\sin(\pi + \alpha) = -\sin\alpha$; $\cos(\pi - \alpha) = -\cos\alpha$;
$\cos(\pi + \alpha) = -\cos\alpha$; $\tan(\pi - \alpha) = -\tan\alpha$; $\tan(\pi + \alpha) = \tan\alpha$

Some other trigonometric relations used in this course are

$\sin 2\alpha = 2\sin\alpha * \cos\alpha$ $\qquad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$\sin^2\alpha + \cos^2\alpha = 1$; $\sin(\alpha \pm \beta) = \sin\alpha * \cos\beta \pm \cos\alpha * \sin\beta$

$\sin\alpha + \sin\beta = 2\sin[(\alpha + \beta)/2] * \cos[(\alpha - \beta)/2]$

$\sin\alpha * \sin\beta = 1/2[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$

The figure 6 presents the graphs of trig. functions for $0 < \alpha < 2\pi$. Note that while the functions ***sine(α) and cos(α) have 2π- period*** and oscillate ***between –1 and +1***, the function ***tan(α)*** (or tgα) ***and cotan(α) have π- period*** and oscillate ***between -∞ and +∞*** .

One has defined the ***inverse*** trig. functions only in ***restricted domains*** so that these function provide a ***single value***.

Function (y =...)	Domain(θ-values)	Range(y-values)
$\sin^{-1}(\theta)$	[-1,1]	[-π/2, π/2]
$\cos^{-1}(\theta)$	[-1,1]	[0, π]
$\tan^{-1}(\theta)$	[-∞,+∞]	[-π/2, π/2]
$\text{cotan}^{-1}(\theta)$	[-∞,+∞]	[0, π]

So, remember that, when using a calculator to get the values for the inverse trig. functions, it will provide results that fall in the ***range presented in this table***. Note that, if you are looking for an angle outside those ranges you will have to convert the result to the corresponding quadrant by referring to the trigonometric circle (see fig.7).

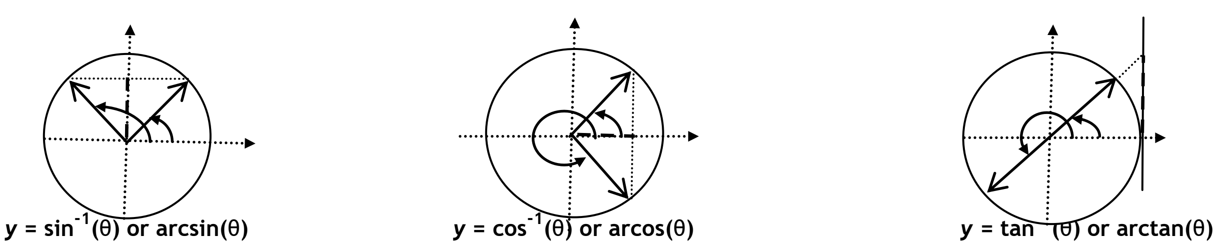


Fig.7 Two different angles provide the same value of a trigonometric function

Sometimes, one may use *small angle approximations* "*S.A.App*.". This means that for small angles, one may substitute the "$sin\theta$" or "$tan\theta$" of an angle to its value ***in radians***; i.e. $sin\theta \cong tan\theta \cong \theta$. One may show that it is true till *second decimal* for $\theta \leq 18^0$. One may use it even for larger angles by referring to *first decimal*.

GRAPHS OF HARMONIC FUNCTIONS

As we will see, the *differential equation* $\frac{d^2y}{dx^2} = -\omega^2 y$ is the ***basic equation*** for the study of oscillations. The solutions of this equation have the form $y(t) = Asin(\omega t + \varphi)$ or $y(t) = Acos(\omega t + \varphi)$

These functions are known as ***harmonic functions***. In general, one may express the *quantity inside brackets* in ***degrees*** or in ***radians***; in this course, one will use always " **radian unit** " for this quantity (i.e. for " $\omega t+\varphi$ ").

A] "HOW TO DRAW THE GRAPH OF AN HARMONIC FUNCTION " - GENERAL RULES

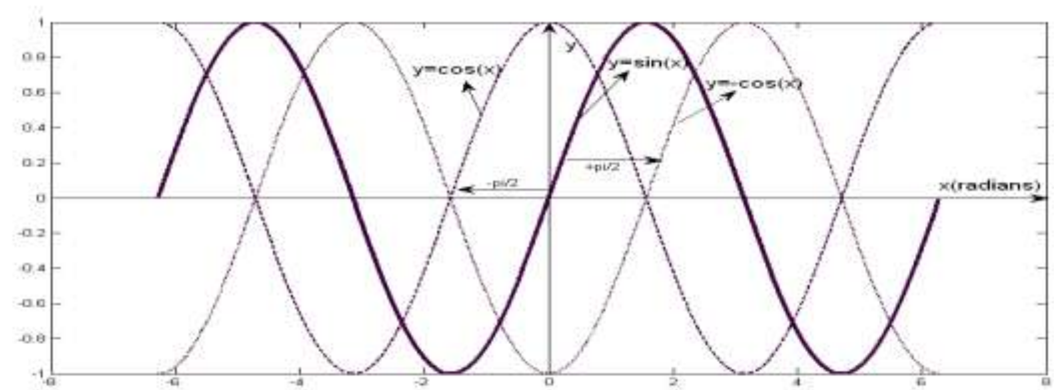
A.1 THE GRAPH OF HARMONIC FUNCTIONS WHEN THERE IS A ***CONSTANT SHIFT***

By following the algebraic values of "PM" and "OP" in the four quadrants of trigonometric circle (fig.5) one can draw the graphs of functions ***sin(x)***, ***cos(x)***, ***- cos(x)*** presented in fig.8.

The figure 8 shows that *a " **left shift by π/2** " of* **sin(x)** *graph* produces the *graph* of the *function* ***cos(x).***
As " ***cos(x) = sin (x + π/2) "***, one finds out that the graph of function ***sin (x + π/2)*** *can be produced by a* ***" π/2 left shift "*** of the ***sin(x)*** graph***.*** *Also, the "**right shift by π/2** "of the* ***sin(x)*** *graph fits to the graph* of function ***"- cos(x)"***. As ***"- cos(x) = sin (x - π/2")***, one finds out that *the graph of function* ***sin (x - π/2)*** *can be produced by a "**π/2 right shift** " of* ***sin(x)*** *graph***.** Those findings are particular cases of the two general rules:

Rule 1. If one knows the **graph** of the ***function*** $y_1 = f(x)$, one can get the **graph** of ***function*** $y_2 = f(x + x_o)$ by a ***left shift*** by " x_o" of the ***graph*** of function $y_1 = f(x)$.
Rule 2. If one knows the **graph** of the ***function*** $y_1 = f(x)$, one can get the **graph** of ***function*** $y_2 = f(x - x_o)$ by a ***right shift*** by " x_o" of the ***graph*** of function $y_1 = f(x)$.

Fig.8



A.2 THE GRAPHS OF HARMONIC FUNCTIONS WHEN THERE IS A **COMPRESSION OR EXTENSION** OF VARIABLE

The graph ***y = sin(x)*** in fig.8 may be seen as a particular case of general form ***y = Asin(bx)*** with "unit *amplitude factor*" (**A=1**) and "unit *variable factor*"(***b = 1***). If ***A < 1*** the amplitude *is compressed (or extended for* ***A > 1****)* and one may figure out easily that the ***graph is compressed (****or extended****) along Oy axis.***
Meanwhile, if the *variable factor* ***b > 1,*** the *graph is* ***compressed along Ox*** axe and if ***b < 1*** the *graph is* ***extended along Ox*** axe. Let's illustrate those two cases by referring to the following example.

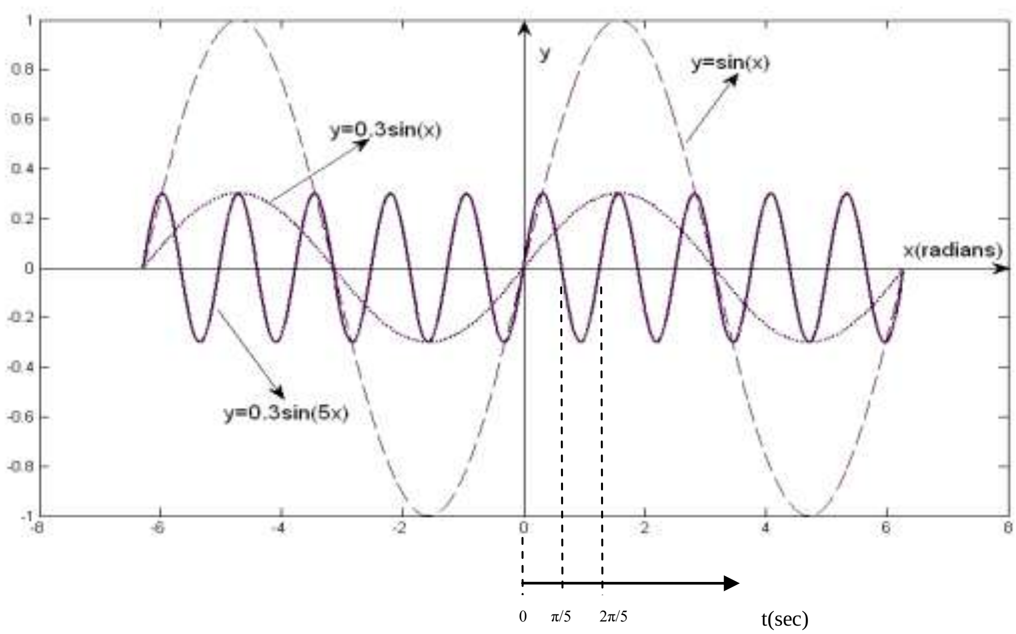
Ex. *Draw the graph of function* ***y[****cm****] = 0.3sin(5t)*** *where* **t** *is time in [sec]*; s*o, A=0.3[cm] and b = 5[r/sec].*
One ***starts by noting 5t = x*** and draws the graph of function ***y_1 = 0.3sin(x)*** (see fig 9) where ***x is in radians.***
Next, one ***converts*** the ***x-axe*** to ***t-axe by dividing all values by 5(t = x/5)*** and ***converting the units to seconds.***
Note that:

1. The last operation depends on ***the unit [r/sec]*** of the *b-parameter* with value "***5".***

2. As the ***period*** of function ***sin(x) is 2π,*** the ***period*** *of function* ***y = 0.3sin5t*** is ***P = 2π/b = 2π/5 = 1.256s.***
Rule: ***<u>the relation P = 2π/ b holds for any argument factor b (bigger or smaller than 1).</u>***

3. If ***b*** > *1* the graph of function **sin(bx)** get ***"compressed, P < 6.28".*** If ***b < 1 it*** get ***"extended, P > 6.28".***

Fig.9



B] DRAWING THE GRAPH OF AN HARMONIC FUNCTION FOR ONE PERIOD

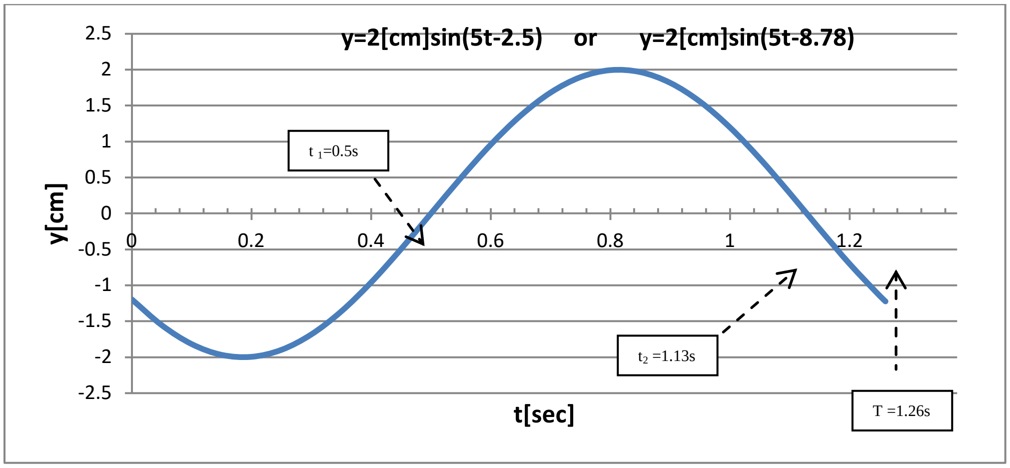
Draw the graph of function ***y = 2***[cm] ***sin(5t***[s] ***– 8.78)***

- At first, one **notes** the ***units*** for "y" as **[**cm**]**; for "5" as **[**r/s**]** and for "8.78" as [radians].
- Next, one check if the magnitude of " *φ* " (i.e. ***8.78*** [r] **)** is ***larger*** than ***6.28*** [r] (which is the ***period of harmonic functions****(sine or cosine*). If this is the case, one notes that the ***shift by a full number of periods does not change the shape of the graph.***
- **Then, o**ne calculates the ***"corresponding difference" that brings to a constant smaller than 6.28 rads*** *(in this case* ***8.78-6.28 = 2.5***[r]) and draws the graph of the ***function y*** = ***2***[cm] ***sin(5t***[s] ***- 2.5)***.
- To draw the graph of this function *for one period* of oscillation, one notes that:
  - the amplitude is $\mathbf{y_{max}}$**= +2cm**; the period is (*see point 2 at A2*) **T**=2π/5=**1.26sec**

- the y-value at t=0**[s]** is y(0)= 2sin(-2.5r) =-1.20[cm]
- the derivative dy/dt at t= 0**[s]**; i.e. y'=10[cm/s] cos(5*0-2.5)=-8[cm/s] **<u>is negative;</u>** i.e. "*motion vs. -oy*".
- one finds two times when ***y=0*** in a period by using conditions ***5t-2.5=0 ..t_1=0.5s ; 5t-2.5=π ..t_2=1.13s.***

- Keeping in mind the upper information, one draws the graph of function ***y = 2***[cm] ***sin(5t***[s] ***– 2.5)*** for one period as shown below. The graph of the function ***y = 2***[cm]***sin(5t***[s] ***– 8.78)*** is the same as this one.

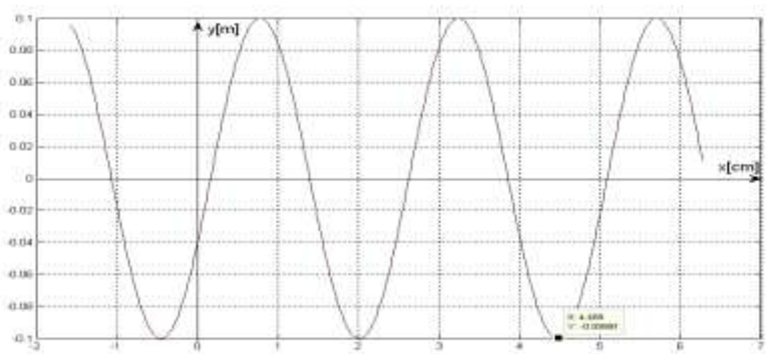
Fig.10



C] FINDING THE ANALYTICAL EXPRESSION OF FUNCTION FOR A GIVEN HARMONIC GRAPH

Find the expression for the graph given in the figure.

Fig.11



The figure 11 presents the graph of an harmonic function(*sine or cosine*). As each form of them can be transformed to the other by a π/2 shift, one may refer to each one. In this course one uses the ***sin function*** form. So, theanalytic form of function that corresponds to this harmonic graph should be $y(x) = A\sin\,(bx + c)$ where the units of each parameter are (read the graph) y[*m*], **A**[***m***], **x**[***cm***], **c**[***r***] and d **b unit** is [***r/cm***]. We have to find the values for each of parameters ***A, b, c.***

From the graph, we find: A = 0.1[m] and "*period" is* ***P = 4.5 - 2 = 2.5 [cm].*** One can find b-parameter by using relation ***b = 2π / P = 6.28 / 2.5 = 2.51.*** So, we get $y(x) = 0.1[m]\sin\,(2.51\left[\frac{r}{cm}\right] * x[cm] + c)$

To find **c** - value, we require that this function produces a given value on the graph; we may refer to ***x = 0*** cm where the graph shows a value ***y = - 0.04[m]***. So, the function must give $y(0) = -0.04[m] = 0.1[m]\sin(c)$

This means that ***sin(c)*** = - 0.4 and c = ***arcsin***(- 0.4) = -0.41r . By checking the other angle that produces the same sine value "i.e. 3.14+ 0.41 = 3.57r" and looking at graph evolution around x = 0 it comes out that the right phase constant is c = - 0.41r. So, $y(x) = 0.1[m]\sin\,(2.51\left[\frac{r}{cm}\right] * x[cm] - 0.41)$