4.1 TRAVELLING WAVES

- We saw the *graphic* presentation of a pulse propagation as a TW wave along a string. Now, we will find the *mathematical* description of this phenomenon. Let's start by drawing Ox axe parallel to the string, assign x = 0 at its free end and take t = 0 at the moment when the end point of pulse shape (i.e. the end point of disturbance) is located at origin (see fig.1). At t = 0 the ***pulse shape*** can be presented by a function ***y = f(x)*** which ***mathematical form*** depends on the shape of the pulse**.** This function defines the ***pulse wave*** in "***space domain"***. To find the function that includes information about the pulse wave in ***time domain*** one:

- considers a pulse with known *shape* function, i.e. known function ***f = f(x)***[1] .
- notes that a " ***feature*** f_1 ***"*** on its shape is ***defined*** by a <u>*given*</u> *value* of ***phase "***x_1***"***.
- considers that the pulse propagates ***without deformation***, i.e. each pulse feature (and its corresponding **<u>phase</u>**) propagates at the <u>*same speed*</u> "**V**" along the string.
- expresses mathematically that the ***f- feature*** of pulse at "***x"*** and time " ***t "*** i.e. ***f(x, t)*** is the same as f_1***-feature*** of pulse at time ***t =*** 0 which is located at $x_1 = x - V*t$ **.**

Next, knowing the shape of pulse wave at t = 0, i.e. its " ***space domain*** " function ***f(x)***, one can get the ***f*** - value at position " ***x "*** at time " ***t " i.e. f(x,t)*** from the relation

$$f(x, t) = f(x_1) = f(x - V*t) \qquad (1)$$

if the pulse is travelling in *positive sense* of x-axis. If the pulse is travelling in *negative sense* of x-axis (see fig.2), one get the expression

$$f(x, t) = f(x_1) = f(x + V*t) \qquad (2)$$

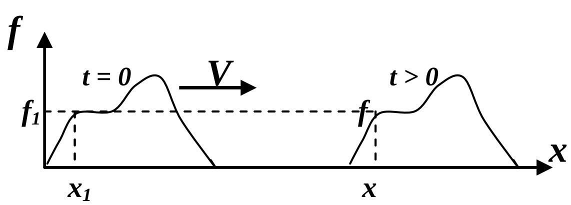


Figure 1 Wave travelling along +Ox sense

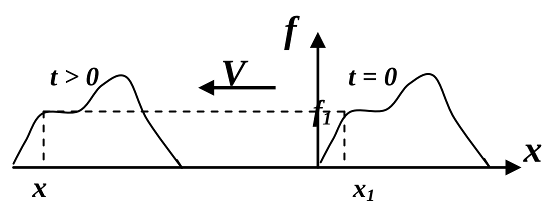


Figure 2 Wave travelling in negative sense of Ox

- A given feature " f_1 " of pulse at **t = 0** corresponds to a given ***phase*** *(via* x_1 *- value)* at the ***space function f(x)***. Meanwhile, the ***travelling wave*** function ***f(x,t)*** has the ***same feature*** (i.e. "$f \equiv f_1$") at all the points (***x***) and the times (***t***) that fulfil the condition

$$(x - V*t) = x_1 \text{ (constant)} \quad (3) \qquad \text{or} \qquad (x + V*t) = x_1 \text{ (constant)} \quad (4)$$

The factor in brackets i.e. (***x +/- V*t***) presents the ***<u>phase of traveling wave</u> function*** .

The derivative of expressions (3, 4) gives

$$dx/dt = +/- V \qquad (5)$$

V- is called "***wave speed"*** and noted as "V_w", but strictly speaking, it is the ***phase speed***.

[1] Actually *f*(x)=*f*(x,0) and the pulse may have any kind of shape. From now on, when referring to a travelling wave phenomena we will call ***phase*** *the* ***quantity inside the brackets*** *of the* ***function that describes the wave***.

- From the practical point of view, one may get the function of a ***travelling wave*** simply by adding the factor " $\pm V^*t$ " after *x*-argument inside the space domain (i.e.shape) function f(x). <u>*Example*</u>: If the *space function* is $f(x) = \sin(2x^2)$ one gets $f_+(x,t) = \sin[2(x-Vt)^2]$ for the function of a pulse wave travelling versus the positive direction and $f_-(x,t) = \sin[2(x+Vt)^2]$ for the function of a pulse wave travelling the versus negative direction of Ox axis.
To check if a given function *may* represent a ***travelling wave***, one should transform it so that time and position appear as <u>*only* one type</u> of factors $(x - V^*t)$ <u>*or*</u> $(x + V^*t)$.

4.2 TRAVELLING HARMONIC WAVES

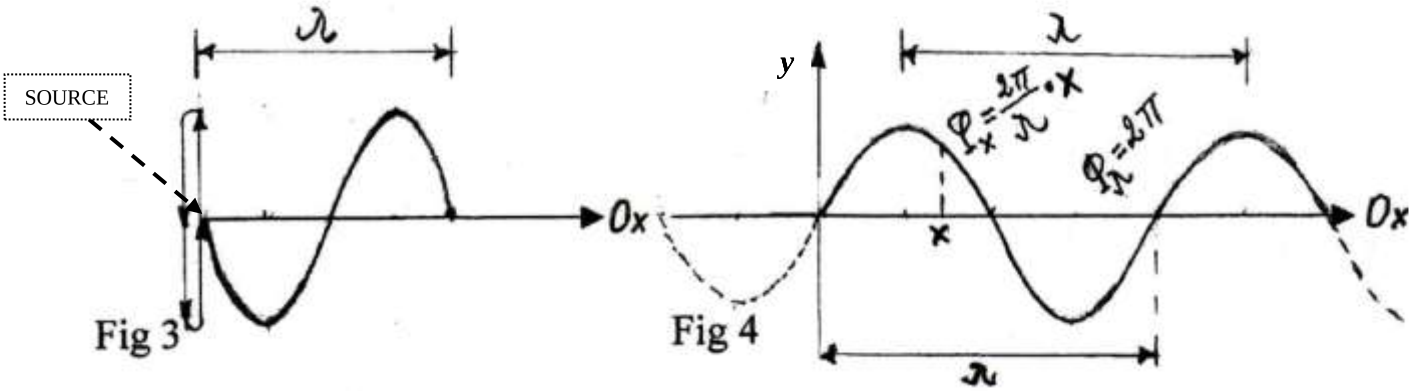
- Consider a string under tension and assume that a device moves up and down one of its ends in a *harmonic way* (i.e. SHM motion). The external device at "***source***" of this TW that travels along the string can change the period of oscillations of the "source". Let " T " be the period of oscillations of the source (*remember that its natural frequency is f = 1 / T*).

-At the moment t = ***T*** (one period) the end point of string (*i.e. the source of wave*) has completed one oscillation and is ready to start the second oscillation. Meanwhile, the disturbance of this wave is propagated through the string till a point at **distance** λ where

$$\lambda = V_w{}^*T \qquad (6)$$

"V_w" stands for the speed of wave travelling through string.

Later on, the string particle at this point oscillates, all time, in the same way as the point of string where the wave is born i.e. "at source" (fig.3). The **length λ** is called " ***wavelength*** " and it is in the role of "*the period* " for this harmonic function f(x) of space coordinate *x*. Note that any two point of string at distance **λ** between them <u>*oscillate with phase difference*</u> <u>***2π***</u> (see fig.4). One says that two points of propagating medium oscillate "***in phase***" if the phase shift between their phasors is *all time constant* and a *multiple of 2π* (i.e. 2π, 4π, 6π,..).



- While the end point of string oscillates as a SHM due to the source motion, this oscillation travels along the string. By taking a string snapshot at a moment t >>T one may see that the string has a sinusoidal shape(fig.4). Next, one can consider this moment as ***t = 0*** and remember that such shapes are described by an harmonic function of form

$$y(x) = A\sin(kx+\varphi_o) \qquad (7)$$

the **phase** " $\boldsymbol{\varphi(x)} = kx+ \varphi_o$ " of this function depends on location "***x***".

Ignoring the source location[2], one can fix the origin of Ox axis at a point with zero "***displacement*** " such that $\boldsymbol{y(0) = A\sin(\varphi_0) = 0,\ \varphi_0 = 0}$ and get $\boldsymbol{y(x) = A\sin(kx)}$.
*The constant **k** has the role of the constant b = 2π / P in the section about the graphs.*
As $P = \lambda$, one gets $\mathbf{k = 2\pi / \lambda}$ and this brings to phase expression $\boldsymbol{\varphi(x)} = \boldsymbol{(2\pi/\lambda)*x}$
So, the wave *shape* or the *space function* that describes the snapshot in figure 4 is

$$y(x,0) = y(x) = A\sin(kx) = A\sin(2\pi/\lambda*x) \qquad (8)$$

where $\boldsymbol{k = 2\pi/\lambda}$ is known as the ***wave number*** (or magnitude of ***wave vector***)

-The three physical quantities $\boldsymbol{k, \omega, V_w}$ are related by the following expressions

$$\boldsymbol{k} = 2\pi/\lambda = 2\pi/(V_w*T) = \boldsymbol{\omega / V_w} \qquad (9) \qquad \text{and} \qquad \boldsymbol{\omega = k * V_w} \qquad (10)$$

- Finally, one uses the *phase shift rule* to get the ***function of travelling waves***.
So, for the propagation of an harmonic wave along the positive sense of axis, one gets

$$y_+(x, t) = A\sin[k*(x-V_w t)] = A\sin(kx - \omega t) \qquad (11)$$

and for propagation of harmonic wave along the negative sense of axis one gets

$$y_-(x, t) = A\sin[k*(x + V_w t)] = A\sin(kx + \omega t) \qquad (12)$$

When considering the presence of an initial phase constant φ_0, these functions become

x-positive propagation $\qquad \boldsymbol{y_+(x, t) = A\sin(kx - \omega t + \varphi_0)} \qquad (13)$

x- negative propagation $\qquad \boldsymbol{y_-(x, t) = A\sin(kx + \omega t + \varphi_0)} \qquad (14)$

-The characteristics of SHM ***motion*** (i.e. $\boldsymbol{v(x)} = \boldsymbol{y'}$ and $\boldsymbol{a(x)} = y''$) for a the ***particle*** of string located at position ' x ' are found from the derivatives of wave function $\boldsymbol{y(x,t)}$ at ***that point***.

[2] We have only a snapshot and we do not know where the source is and what direction the wave propagates.

4.3 STANDING WAVES

- Let's consider two harmonic TW waves <u>propagating</u> along opposite senses on the same string. Assume that they have <u>equal amplitudes</u>, same <u>frequency</u> and $\varphi_0 = 0$. Then, the function of the wave travelling on negative direction is : $y_- = A\sin(kx + \omega t)$ (15)

function of the wave travelling on positive direction is : $y_+ = A\sin(kx - \omega t)$ (16)

- Superposition principle: the ***disturbance*** (*displacement in space*) at any string particle is

$$y = y_- + y_+ \qquad (17)$$

$$y = y_- + y_+ = Asin(kx + \omega t) + Asin(kx - \omega t) = \mathbf{2Asin(kx)cos\ (\omega t)} \qquad (18)$$

Important: In this wave function, the " ***space*** " *part* of phase and the " ***time*** " *part* of phase are ***separated***. This type of function describes a <u>***standing wave***</u> .

-The " ***space*** " function ***sin(k*x)*** defines the positions of "***nodes***" (*y(x) = 0 all time*) at the locations where $\boldsymbol{\varphi(x) = k*x = (+/-)\ n*\pi}$. *n=0,1,2,3,…*

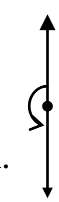
As the *phase shift* between two ***consecutive*** nodes is $\boldsymbol{\pi}$, their distance $\boldsymbol{\Delta x}$ on string is

$$\Delta\varphi = k\Delta x = \frac{2\pi}{\lambda}\Delta x = \pi \to\to\to \Delta x = \frac{\lambda}{2}$$

Also, the function ***sin(kx)*** defines the locations of "anti***nodes***" (oscillations at maximum amplitude 2A). At an ***anti-node, sin(k*x) = +/-1*** and its location is found by the condition

$$\boldsymbol{\varphi(x) = k*x = +/-\ (2n+1)*\pi/2}. \quad n=0,1,2,3,\ldots$$

The distance between two ***consecutive*** antinodes is $\frac{2\pi}{\lambda}\Delta x = \pi \to\to \Delta x = \frac{\lambda}{2}$



One may figure out (fig. 5,6) that there is an *anti-node* between *two nodes* and vice-versa.

-The quantity $A(x) = 2A\sin(kx)$ in (18) is the ***amplitude*** of oscillations ***at location "x".*** In a ***standing wave*** **each point** "***x***" of medium oscillates at "***its own amplitude A(x)***" while in a ***traveling wave***, **all points** of medium oscillate at ***same amplitude A***. In both cases, the string particles oscillate around equilibrium position at same frequency " $\boldsymbol{f = \omega/2\pi}$ " (as ***built by the source***). <u>**Note:**</u> Standing waves do not travel. Travelling waves do travel.

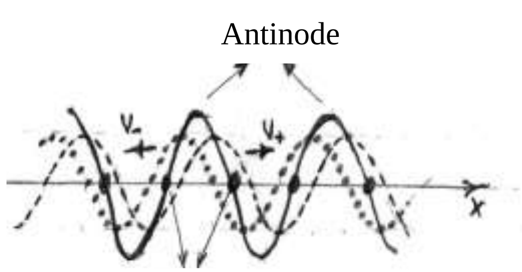
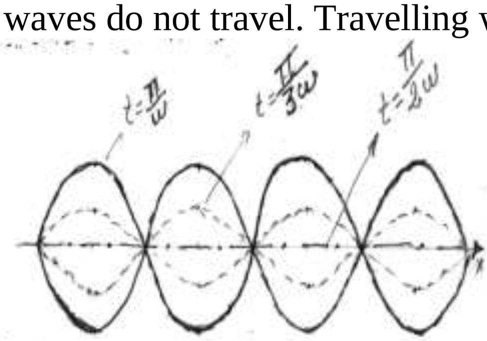


Fig 5



Standing Wave

Fig 6

4.4 RESONANCE

- The function $y = 2A\sin(kx)\cos(\omega t)$ does ***not*** have any ***restriction*** for the ***frequency*** or ***wavelength*** of a standing wave if there is not any ***boundary restriction***. But, in case of standing waves on a string of finite ***length L*** with a fixed ***end point, this point*** must be all time a ***node*** and the space part of wave function must fulfill the condition

$$\sin(kL) = 0 \to kL = n\pi \to \frac{2\pi}{\lambda} L = n\pi \Rightarrow\Rightarrow\Rightarrow \lambda_n = \frac{2L}{n} \qquad (19)$$

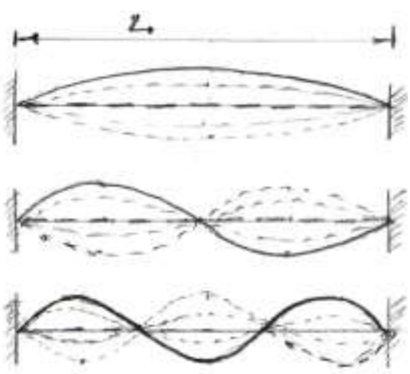
A string with finite ***length L*** and a fixed end cannot "host" *standing waves* for any value of λ. It "*allows*" only those that fulfill the condition $\lambda_n = \frac{2L}{n}$ n=1,2,3,…

-The corresponding frequencies are known as "***resonance frequencies***" of string.

As, $$\lambda_n = \frac{2L}{n} \to V_w * T_n = \frac{2L}{n} \to \frac{V_w}{f_n} = \frac{2L}{n} \to f_n = n\frac{V_w}{2L} \qquad (20)$$

The source of waves can oscillate at different frequencies and waves will propagate on string. But***, only if the source frequency equals one of values defined by condition*** (20) ***a standing wave appears on the string.*** One says that the ***system source-string*** is vibrating ***"in resonance*** " or ***a resonance is produced in the string*** (*propagating medium*).

- Note that the resonance frequencies are " ***string characteristics*** ".
The first one, i.e. $f_1 = V_w/2L$ is known as ***fundamental*** or ***first harmonic frequency***.
The second one, $f_2 = 2 * V_w/2L$ is labelled as ***second harmonic frequency*** and so on.
The ***boundary condition*** (*node at location equal to* the ***length of string L***) ***defines*** the frequencies of *resonant modes* or *normal modes of the string*. Note that this remains true for any oscillating system; ***the frequencies and wavelengths of resonant modes of any oscillating system are defined by its boundary conditions***.



L=λ/2 ;n=1
Fundamental mode

L= λ ; n=2
2nd Harmonic

L= 3λ/2 ; n=3
3rd Harmonic

Fig. 7

Note:
These kinds of string vibration are called "***normal modes of the string***".

$$f_n = n\frac{V_w}{2L}; \quad n = 1,2, \ldots$$

$$\lambda_n = \frac{2L}{n}; \_ n = 1,2,..$$