REMEMBER

-The general form of the **wave equation** $\frac{\partial^2 y}{\partial x^2} = \frac{1}{\upsilon^2}\frac{\partial^2 y}{\partial t^2}$ is written as $\frac{\partial^2 E}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$ for light (*and all E&M*) waves; the electric field value " *E* " stands for the " *displacement* " and **c ≈ 3*10^{8}m/s** (*299792458m/s*).

-The wave model requires a medium for propagation of light waves and one labelled this medium as "aether".
A frame tied to aether would be the *absolute reference frame* for all laws of physics.
Michelson and Morley used the earth motion around the sun and the interferometer of Michelson to check the existence of ether. They found a negative result; the aether does not exist for real.

-The laws and principles of mechanics are **covariant** versus all inertial frames when using the rules of Galileo transformations. Maxwell theory is not covariant versus inertial frames when using those transformations rules.

- **Special** theory of relativity means that it is valid " **only for inertial frames** ". The first postulate of relativity : ***"All physics laws have the same form in all inertial frames"*** requires ***covariance*** for ***all physics laws***.
<u>*Consequence*</u>: There is no need for an "absolute frame of reference tied to aether". To make this postulate work for the light (and all E&M) waves, one has to use *Lorentz transformations* (see *next sections*) instead of *Galilean transformations*. One may show that *Lorentz transformations* convert into the *transformations* of *Galileo* for common speeds (*routine human experience, see next sections*).

- The second postulate of relativity: ***The speed of light in vacuum is the same versus all inertial frames of reference (i.e. it is independent on the selection of frame where one measures the light speed).***
<u>*Consequence:*</u> ***An object with mass*** *(material particle or observer)* ***cannot travel at speed c. The speed of light in vacuum is a maximum unattainable by any object with mass.***

- A physical measurement is an *event* with coordinates (x, y, z, t) in a frame S. The *same event* has the coordinates (x', y', z', t') in another frame S'. The **rest frame** of an object is the inertial frame versus which this ***object is at rest***.
<u>**The proper length L_0 is the distance between ends of an object measured in object's rest frame; *it is the largest L-value***</u> one can get when measuring the length of this object.
The length contraction concerns the dimension of an object *along the direction of its motion;* its dimensions <u>*perpendicular to direction of motion do not change*</u>. The contracted length of object

is given by expression $L = \frac{L_0}{\gamma}$ where $\gamma = \frac{1}{\sqrt{\left(1-\frac{\upsilon^2}{c^2}\right)}} = \frac{1}{\sqrt{\left(1-\beta^2\right)}}$ and $\beta = \frac{\upsilon}{c}$

- Here are some values of γ (**1 ≤ γ < + ∞**) for different speeds (**0 ≤ β <1**)

υ / c = β	0.001	0.01	0.1	0.6	0.8	0.98	0.995	0.9965	0.9992
γ	1.0000005	1.00005	1.005	1.25	1.67	5	10	12	25

One may see that relativistic effects become "***noticeable at third decimal"*** for speeds of order ***10^{7}m/s*** *(β=0.1)*.

-The ***interval of time*** between two events ***depends on the frame*** where it is measured. If the *events happen at different locations,* one needs **two watches** to measure the time interval between them. But, there is **one frame** where the two ***events happen at same location*** and a *single* ***watch*** can measure the time interval. At this frame **(rest frame of this watch),** the **interval of time between the two events "Δt_0" is the shortest one**. The time interval between the two same events at any other frame is larger (*time dilatation*) **$\Delta t = \gamma\Delta t_0$.**

1.RELATIVISTIC DOPPLER EFFECT

-Consider a ***monochromatic light wave*** emitted by a source in front of a train moving at very high speed "$\boldsymbol{v}$" versus an inertial frame **S** (fig.1) tied to station. The observers O (at *rest in station*) and O' (*at rest in train frame **S'***) ***measure the period*** of this wave and next ***calculate*** the wave frequency as "*f = 1/T*". Being at rest versus "*light source*", the observer O' measures time interval between emissions of t***wo consecutive*** *wave* ***crests*** by **one watch** at the **same place; so,** he gets the **proper period T_0.** Next, he **calculates** the **proper (**or natural**) frequency** of the wave in **frame O'** as $f_0 = \frac{1}{T_0}$ (1)

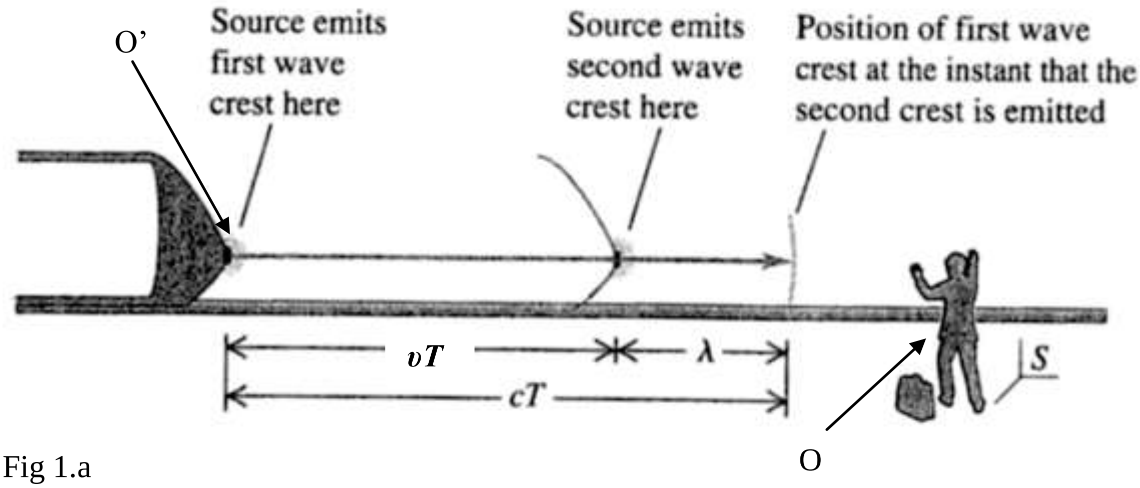


Fig 1.a

-Two observers O in frame S use two watches to ***measure*** the period ***T*** (*not proper* time) between the emission of two consecutive wave crests. For them, during the time ***T***, the train has advanced by "$\boldsymbol{v*T}$" while the first wave crest has advanced by "***c*T***". So, they express the wavelength "$\boldsymbol{\lambda}$" (in S frame, see fig 1.a) as *distance between* the source locations at the moments it emits the two consecutive crests.

$\lambda = c*T - \upsilon*T = (c-\upsilon)*T$ (2) and they calculate the (not proper) period $T = \frac{\lambda}{c-v}$ (3)

As $T = \gamma T_0$ one finds out that $\frac{\lambda}{c-v} = \gamma T_0$ (3')

By substituting at (3') $\boldsymbol{\lambda}$ = c * T one get $\frac{c*T}{c-v} = \gamma T_0$ and $T = \frac{c-v}{c} * \gamma T_0$

Then, $T = \frac{c-v}{c} * \frac{1}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} T_0 = \frac{c-v}{c} * \frac{1}{\left(\frac{c^2-v^2}{c^2}\right)^{1/2}} T_0 = \frac{(c-v)}{\frac{c}{c}} * \frac{(c^2-v^2)^{1/2}}{(c^2-v^2)} T_0$ and

$$T = (c-v) * \frac{(c^2-v^2)^{\frac{1}{2}}}{(c^2-v^2)} T_0 = (c-v) * \frac{(c^2-v^2)^{\frac{1}{2}}}{(c-v)(c+v)} T_0 = \frac{(c^2-v^2)^{\frac{1}{2}}}{(c+v)} T_0 = \frac{(c-v)^{\frac{1}{2}}(c+v)^{\frac{1}{2}}}{(c+v)} T_0$$

Finally, $T = \frac{(c-v)^{\frac{1}{2}}}{(c+v)^{\frac{1}{2}}} T_0$ or $\frac{1}{f} = \frac{(c-v)^{\frac{1}{2}}}{(c+v)^{\frac{1}{2}}} \frac{1}{f_0}$ which brings to $f = \frac{(c+v)^{1/2}}{(c-v)^{1/2}} * f_0$ (4)

So, the frequency "f" of a light wave, as measured by observers O at rest in frame S, is related to the natural frequency "f_0" of source (and emitted light wave) moving at speed "$\boldsymbol{v}$" versus frame S as follows

$$f = \sqrt{\frac{c+\upsilon}{c-\upsilon}} * f_0 \qquad (5)$$

- Note that this expression contains only the *relative speed* "***υ***" of source *versus the observers* O, not its speed versus propagating medium. This avoids the transporting medium for light (**no need for ether**).
At Doppler effect for sound, one had to deal with two velocities versus propagating medium ($\boldsymbol{V_s}$ and $\boldsymbol{V_O}$) because there is a medium that propagates the sound wave (ex. *air, water...*).

- The expression (5) corresponds to the case when the source approaches the observer. One may easily find out that when the source ***moves away*** from the observer (see fig 1.b)

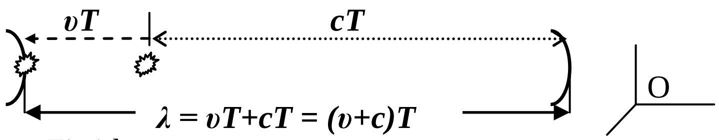


Fig1.b

$$f = \sqrt{\frac{c-\upsilon}{c+\upsilon}} * f_0 \qquad (6)$$

From (5) and (6), one may figure out that the observer O measures $f > f_0$ if the ***source*** of light is ***approaching*** him and $f < f_0$ if the ***source*** of light is ***moving away*** from him.

- If the speed of light source vs. O-observer is $\upsilon << c$ (i.e. $\beta = \upsilon/c << 1$) one get a simple expression for the ***Doppler shift*** " $\Delta f = f - f_o$ " of recorded light ***frequency***. Starting from expression (5)

$$f - f_0 = \left(\sqrt{\frac{c+v}{c-v}} - 1\right) * f_0 \rightarrow \frac{\Delta f}{f_0} = \frac{(c+v)^{\frac{1}{2}} - (c-v)^{1/2}}{(c-v)^{1/2}} = \frac{(1+v/c)^{\frac{1}{2}} - (1-v/c)^{1/2}}{(1-v/c)^{1/2}} \cong \frac{1+\frac{1}{2}*\frac{v}{c} - 1 + \frac{1}{2}*\frac{v}{c}}{1-\frac{1}{2}*\frac{v}{c}} = \frac{\frac{v}{c}}{1-\frac{1}{2}*\frac{v}{c}} \cong \frac{v}{c}$$

and finally

$$\frac{\Delta f}{f_0} = \pm \frac{\upsilon}{c} \qquad \text{(+ for approach; - move away)} \qquad (7)$$

This expression allows to calculate the Doppler <u>shift of frequency</u> for common light sources in motion versus observer. The frequencies of light waves are very high and are not measured easy way at a good precision, but one can measure light wavelength at very good precision by interference methods.
So, one converts the expression (7) by using the basic relationship $\boldsymbol{\lambda = c / f}$.

As the differential $d\lambda = c*(-df / f^2)$, one gets $\dfrac{d\lambda}{\lambda} = \dfrac{c*(-df / f^2)}{c / f} = -\dfrac{df}{f}$.

For " measurable " changes $\dfrac{\Delta\lambda}{\lambda_0} = -\dfrac{\Delta f}{f_0}$ i.e. $\dfrac{\Delta\lambda}{\lambda_0} = -(\pm)\dfrac{\upsilon}{c}$ and finally

$$\frac{\lambda - \lambda_0}{\lambda_0} = \mp \frac{v}{c} \qquad \textbf{(- for approach; + move away)} \qquad (8)$$

So, there is a shift *versus the blue* (*recorded* $\lambda < \lambda_0$) if the source is *approaching observer* and a shift *versus the red* (*recorded* $\lambda > \lambda_0$) of wavelength if the source is *moving away* from the observer.
By using this expression in astronomy, one has measured the velocities of galaxies and has found that they are moving away from the earth. Those measurements confirm the " expansion" of universe. Also, one uses the expression (8) to measure the velocity of objects on the earth (ex. the Police handheld radar).

2.LORENTS TRANSFORMATIONS OF COORDINATES BETWEEN INERTIAL FRAMES

-As mentioned previously, the Galileo transformations (*between two inertial frames*) $x' = x - \upsilon * t$; $t' = t$ do not accept the ***second postulate*** of relativity. Also, the Maxwell theory is not covariant with respect to Galileo transformations. For these reason, Einstein decided to use ***Lorentz transformations*** for relation between coordinates of same event in two different inertial frames. This type of transformations allows the second postulate of relativity and make Maxwell equations covariant.

-Consider two inertial frames (fig. 2) S and S'(*moving along Ox axis of S at velocity "+ υ"*). An ***event*** has coordinates (***x, y, z, t***) in frame ***S*** and coordinates (***x', y', z', t'***) in frame ***S'***. Note that, there is changes of time and lengths along Ox axis but there is ***no length changes*** on the directions ***perpendicular to Ox;*** i.e. $x' \neq x$; $t' \neq t$; $y' = y$ and $z' = z$. Lorentz transformations relate the two relativistic coordinates that change (***x', t'***) measured in moving frame *S'* to the corresponding values (***x ; t***) in frame *S* as follows

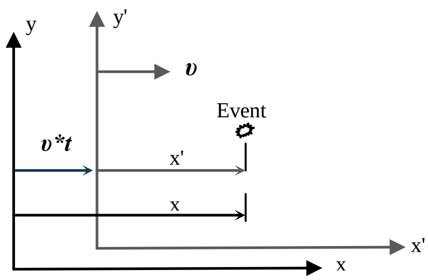


Fig 2

$$x' = \gamma(x - \upsilon * t) \tag{9}$$

$$t' = \gamma(t - \frac{\upsilon}{c^2} * x) \tag{10}$$

- The relation (10) shows clearly that the variable "***t'*** " is not the same as ***"t" and confirms*** that Einstein definition for an event as a set (***x, y, z, t***) is ***essential***. ***In the theory of relativity the time and the space merge together and form a 4D structure.*** When dealing with "*relativistic situations*", the scientists refer to the events as a set of four coordinates (x_1, x_2, x_3, x_4) *without distinguishing the time from space*.

$$x_1 = x; \_\_\_ x_2 = y; \_\_\_ x_3 = z; \_\_\_ x_4 = c * t \tag{11}$$

Within this model, one writes the expressions (9,10) in a same symmetry mathematical form, as follows

$$x_1' = \gamma[x_1 - \frac{\upsilon}{c} * (ct)] = \gamma[x_1 - \beta * x_4]$$
$$x_4' = c * t' = \gamma(c * t - \frac{\upsilon}{c} * x_1) = \gamma(x_4 - \beta * x_1) \tag{12}$$

- Note that the Lorentz transformations (9,10) convert to Galileo transformations for "normal" values of speed because in this case ***υ << c*** , $\upsilon / c^2 = 0$ and ***γ =1 ;*** *then* $x' = x - \upsilon * t$; and $t' = t$.

-Lorentz transformations are fully symmetrical. One can calculate the coordinates in system S from coordinates in system S' by using the same expressions (9,10 or 12). One has only to take into account that velocity of frame S versus S' would be negative *"- υ"* which gives to these transformations the form

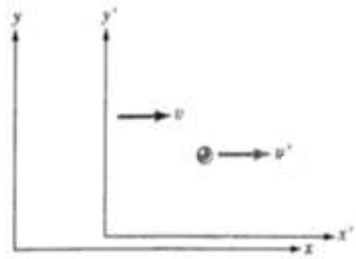
$$x = \gamma(x' + \upsilon * t') \tag{9'}$$

$$t = \gamma(t' + \frac{\upsilon}{c^2} * x') \tag{10'}$$

$$x_1 = \gamma(x_1' + \beta * x_4) \quad \text{and} \quad x_4 = \gamma(x_4' + \beta * x_1) \tag{12'}$$

3.LORENTS TRANSFORMATIONS OF VELOCITIES BETWEEN INERTIAL FRAMES

-Consider an inertial frame **S'** moving at velocity "***+υ***" with respect to frame ***S*** along its Ox axis (fig 3). Next, refer to coordinates of a *particle* moving at velocity ***u*** versus frame **S and *u'*** versus frame **S'.** As in classical mechanics : ***x' = x - υ*t*** and ***t' = t*** (*Galileo transformations*), one can get



$$u' = \frac{dx'}{dt'} = \frac{dx'}{dt} = \frac{d(x-vt)}{dt} = \frac{dx}{dt} - v = u - v \qquad (13)$$

Fig 3

- In special relativity, the time depends on reference frame (i.e. ***t' ≠ t).*** So, one has to proceed following the formal algebra rules to find the expression for the derivative (***u´ = dx´/ dt´***). From expressions (9,10)

$$dx' = \gamma(dx - \upsilon * dt) = \gamma * dt(\frac{dx}{dt} - \upsilon) = \gamma * dt(u - \upsilon) \qquad (14)$$

$$dt' = \gamma(dt - \frac{\upsilon}{c^2} * dx) = \gamma * dt(1 - \frac{\upsilon}{c^2} * \frac{dx}{dt}) = \gamma * dt(1 - \frac{\upsilon}{c^2} * u) \qquad (15)$$

$$u' = \frac{dx'}{dt'} = \frac{\gamma * dt(u - \upsilon)}{\gamma * dt(1 - \frac{\upsilon}{c^2} * u)} = \frac{u - \upsilon}{1 - \frac{\upsilon}{c^2} * u} \qquad (16)$$

-Note that:

a) The expression (16) transforms to classical Galileo transformation (13) of velocities when one deals with "***common*** " velocity values because if ***υ, u << c*** then [1-(υ/c)*(u/c)] = 1.

b) If the "*particle"* is an *electromagnetic photon* its speed versus frame S is ***u = c***. Then, expression (16) gives a result that respects the second postulate of relativity; i.e. speed "c" versus frame S´, too.

$$u' = \frac{c - \upsilon}{1 - \frac{\upsilon}{c^2} * c} = \frac{c - \upsilon}{c - \upsilon} * c = c \qquad (17)$$

4.RELATIVISTIC DYNAMICS

- As shown, the Lorentz transformations fit to the second postulate, i.e. they allow *the same speed of light versus all inertial frames and* the Maxwell theory is covariant with respect to Lorentz transformations(see other sources). Also, Einstein did the necessary update of the Classical Mechanics parameters to make them covariant with respect to Lorentz transformations, too. In the following, one will present the relativistic corrections for three major parameters of dynamics; ***inertial mass, linear momentum and energy***.

- Consider a box with length "L", a light source "S" and a detector "D" at box ends; there is ***vacuum in the box***. Assume that the box *is **at rest*** versus an ***inertial frame Ox*** with its left side located at x = 0 (see fig 4).

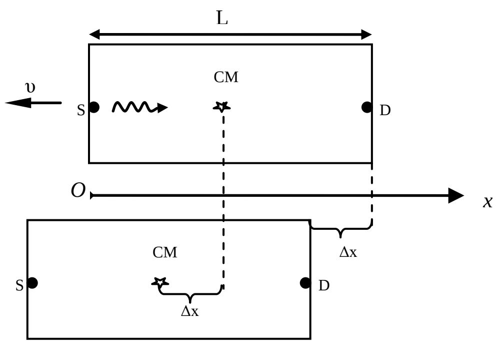


Fig 4

Suppose that (at t=0s) the source S emits a *light pulse* that travels toward detector D. As we know from the travelling wave model, this pulse can be modelled as a *wavelet* that *transports energy* and *linear momentum* from S to D. Maxwell theory shows that, for an E&M wavelet, the ***magnitude "p"*** of linear momentum is related to transported energy as follows

$$p = E/c \qquad (18)$$

E is the energy transported by light pulse.
c is the speed of light.

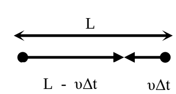
- If the box is an *isolated system,* there is no action from outside. So, $F_{ext} = \Delta P_{sys}/\Delta t = 0$ and this means that the ***linear momentum*** of the ***set box - light pulse*** must remain constant all time; i.e. $P_{sys} = P_{sys(t=0)} = \boldsymbol{0}$***(at rest)***. This requires that, while the light pulse transports the *linear momentum "**p**" right side*, the mass ***M*** of box gets a velocity "***-υ***" directed to the *left side* such that the net linear moment ***of set box-pulse remains zero***.

$$-M * \upsilon + p = 0 \rightarrow M * \upsilon = p = \frac{E}{c} \qquad (19)$$

So, the box velocity should have the magnitude

$$\upsilon = \frac{E}{M * c} \qquad (20)$$

- When the light pulse hits on the detector "D", the center of mass of box (CM) has travelled left side a distance " $\Delta x = \upsilon * \Delta t$ " and the light pulse has traveled the distance $c * \Delta t = L - \Delta x = L - \upsilon * \Delta t$
This means that the wavelet has travelled during the interval of time



" $c * \Delta t + \upsilon * \Delta t = L$; $\Delta t(c + \upsilon) = L$ "

$$\Delta t = \frac{L}{c + \upsilon} = \frac{L}{c + E/Mc} \qquad (21)$$

At the end of this same time, after absorption of light pulse ,the *center of mass* of the box would be shifted (see fig 4) *to the left* by

$$\Delta x = \upsilon * \Delta t = \frac{E}{M * c} * \frac{L}{c + E/Mc} = \frac{EL}{Mc * (Mc^2 + E)/Mc} = \frac{EL}{E + Mc^2} \qquad (22)$$

Once the detector absorbs the light pulse, the box receives a right-side directed *linear momentum* " ***p*** " which cancels out its left - side *linear momentum* ***M*υ*** that the box had until this instant. At this moment, the linear momentum of the box becomes zero (i.e. v = 0) and it stops its motion.

- Then, at the end of process, the ***center of mass of " isolated box** " should be* ***shifted*** *to the left by* $\boldsymbol{\Delta x}$ and *this happens without any action from outside the box*! But, this **contradicts the conservation of linear momentum because as** $P_{sys} = \mathbf{m_{sys}} * \mathbf{v_{sys}} = \mathbf{m_{sys}} * \mathbf{d(X_{cm})/dt} = P_{sys(t=0)} = \mathbf{0}$ ***one get*** $\mathbf{d(X_{cm})/dt} = \mathbf{0}$ ***and*** $\mathbf{X_{cm} = const.}$
Example: " The center of mass of a "set boat & person" at rest on water does not change position when the person walks from one end to the other end of the boat" .
To get rid of this odd situation, one may " ***formally associate*** *to the light pulse a certain amount of virtual* ***mass m*** *such that the* ***center of mass*** *CM of the set box & light pulse* (with virtual mass) *remains at same place during all the time that light pulse travels from S to D.* With this assumption, one requests that the x-coordinate (23) of CM of the set of two masses ***M*** and ***m*** remains all time at same location

$$X_{CM} = \frac{M*x_{Box} + m*x_{pulse}}{M+m} \tag{23}$$

Assuming that box mass *M* is symmetrically distributed, at t = 0 its CM is at x = L/2 and at t = Δt it is at x = L/2-Δx (see fig.4). Mass *m*, at t = 0 is at x = 0 and at t = Δt it is at x = L - Δx (the right side of box). So,

at t = 0 (just after emission) $$X_{CM} = \frac{M*L/2 + m*0}{M+m} = \frac{ML/2}{M+m} \tag{24}$$

at t = Δt (just before absorption) $$X_{CM} = \frac{M*(L/2-\Delta x) + m*(L-\Delta x)}{M+m} \tag{25}$$

By equalizing (24) to (25) , one get out the value of virtual mass ***m*** that would keep CM at same place.

$$\frac{ML}{2} = \frac{ML}{2} - M\Delta x + mL - m\Delta x \quad \text{and} \quad \text{next} \quad m(L-\Delta x) = M\Delta x \quad \text{or} \quad m = \frac{M\Delta x}{L-\Delta x} = M\frac{\Delta x}{L-\Delta x}$$

As $$\frac{\Delta x}{L-\Delta x} = \frac{EL/(E+Mc^2)}{L-EL/(E+Mc^2)} = \frac{EL}{EL+LMc^2-EL} = \frac{E}{Mc^2} \quad \text{one gets} \quad m = M\frac{E}{Mc^2} = \frac{E}{c^2}$$

- This last relation $\boldsymbol{m = E/c^2}$ is written mostly as $$E = mc^2 \tag{26}$$
Let's check if this relation is accepted by the dimensional analysis condition; say in SI system
*[E]=J=N*m=(kg*m/s²)*(m)= kg* m²/s² and [mc²]=kg*m²/s² ..so it does hold.*

Even though one get the relation (26) by referring to a "*virtual inertial mass* ", as it comply to dimensional analysis, it seems to reveal a ***fundamental reciprocal correspondence between the mass and the energy: i.e.*** the ***inertial mass*** of a *material object* ***is proportional to the energy "encapsulated" in object***. Knowing that, due to its kinetic energy, an object with mass in motion has a larger total energy compared to its energy at rest, one may write the first form of relation (26)

for a *particle at rest* as $m_0 = E_0/c^2$ or $E_0 = m_0c^2$ (26')
and for the *same particle in motion* as $m = E/c^2$ or $E = mc^2$ (26")
As $E > E_o$, it comes out that $m > m_o$ i.e. *the inertial mass of a particle increases with increase of its speed.*

- The ***linear momentum*** of a ***relativistic*** particle is written the *same way* (27) as in classical mechanics provided that one uses expression (28) for its ***inertial mass***. As shown in next section, this mass expression allows the ***covariance*** for the principle of linear momentum conservation versus Lorentz transformations.

$$\vec{p} = m\vec{v} \tag{27}$$
$$m = \gamma m_0 = \frac{m_0}{\sqrt{1-(\frac{v}{c})^2}} \tag{28}$$

-The **rest mass m_0** of an object is measured in the **rest frame** *of the object* and the quantity $\boldsymbol{E_o = m_o * c^2}$ is known as its ***rest energy***. As "common objects" move versus the earth at speeds much lower that 10^6m/s ($\beta = 0.01$, $\gamma = 1.00005$ and $m = \gamma\, m_o \cong m_o$) their ***relativistic mass*** is practically equal to their rest mass. This explains why the classic mechanics that considers the mass as a constant provides good results for their motion. If an objects moves very fast (say at speed *over*10^7*m/s*) versus a reference frame, one has to take in consideration the change of its mass. In these situation, one calculates the ***kinetic energy*** of the object as the difference between its ***total energy*** $\boldsymbol{E = m*c^2}$ and its ***rest energy*** $\boldsymbol{E_o = m_o*c^2}$ ***:***

$$K = E - E_0 = mc^2 - m_0 c^2 = (m - m_0) * c^2 = (\gamma - 1) m_0 * c^2 = (\gamma - 1) E_o \qquad (29)$$

Note: Don't use the classic expression $K = mv^2/2$ *in situations where the relativistic effects are noticeable.*

- In classical mechanics, $K \sim \upsilon^2$, $p \sim \upsilon$ and there is no limits for the speed of particles. In special relativity, there is a limit for a particle speed; if ***the speed*** approaches ***"c"*** value***, the kinetic energy*** and ***the linear momentum*** of particle go versus the infinity (see 27, 30 and the graphs of these functions in fig 5,6).

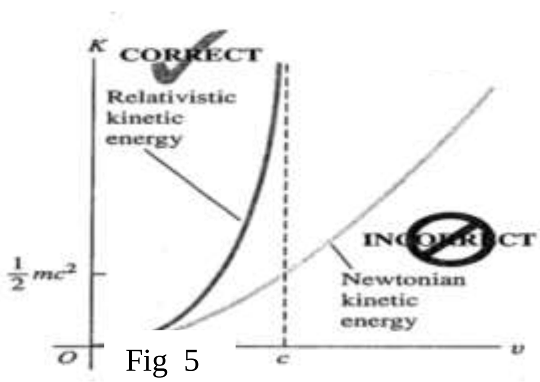


Fig 5

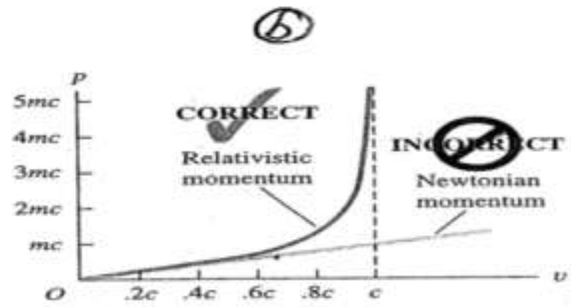


Fig 6

Here one gets once more to the restriction about the maximum speed of objects with mass. *To increase the kinetic energy of an object with mass, one should get work provided by an energy source. As the speed "c" corresponds to infinite kinetic energy, one needs a source that provides infinity amount of work.* As there is no source that contains infinite energy, there is *no way an object with mass* can get a speed equal to "**c**".

The speed of light in vacuum is unattainable by objects with mass.

- The relation between the ***magnitudes of total energy E*** and ***linear momentum p*** is of special interest for different phenomena that happen in a " relativistic model ". By combining the following relations (30, 31)

$$E = m * c^2 = \gamma * m_0 c^2 \qquad (30) \qquad\qquad p = m * \upsilon = \gamma * m_o * \upsilon \qquad (31)$$

in such a way that $\boldsymbol{\upsilon}$ and $\boldsymbol{\gamma}$ can be removed, one may get to the relation (34) as follows:

Write $$\left(\frac{E}{m_0 c^2}\right)^2 = \gamma^2 \quad (32) \qquad \frac{p*c}{m_o c^2} = \frac{\gamma * m_o * \upsilon}{m_o c^2} c = \gamma \frac{\upsilon}{c} \quad \text{and} \quad \left(\frac{p*c}{m_0 c^2}\right)^2 = \gamma^2 \frac{\upsilon^2}{c^2} \qquad (33)$$

Next, by subtracting the relation (33) from the relation in (32) one gets

$$\left(\frac{E}{m_0c^2}\right)^2-\left(\frac{p*c}{m_0c^2}\right)^2=\gamma^2*(1-\frac{\upsilon^2}{c^2})=\frac{1}{(1-\frac{\upsilon^2}{c^2})}*(1-\frac{\upsilon^2}{c^2})=1$$

i.e. $E^2-p^2c^2=(m_0c^2)^2$ and finally $E^2=p^2c^2+(m_0c^2)^2$ or $E^2=(pc)^2+E_0^2$ (34)

Expression (34) relates the ***total energy*** with ***linear momentum*** and the ***rest energy of relativistic objects***. One may remember easily this relation by using the ***energy triangle***. The quantity ***pc*** in this triangle has energy unit. As the *energy units* used in particle physics are ***eV***, ***MeV, GeV***, one uses "energy related" *units* ***eV/c, MeV/c, GeV/c*** for *linear momentum* in this field of physics.

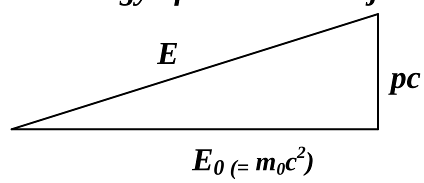


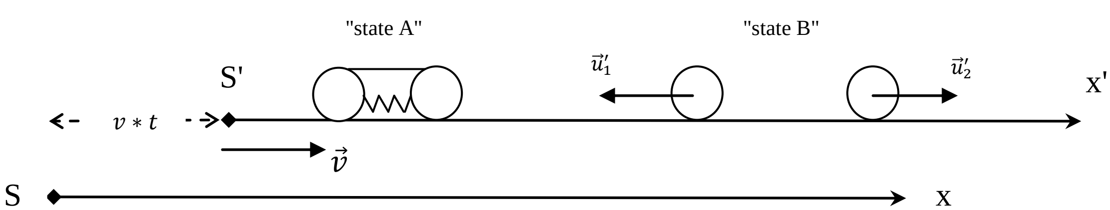
Fig 7 Energy triangle

Note that for:

a) *light or radiation* ***particles,*** $m_o=0$ and this relation produces $E=p*c$ (as given by Maxwell theory).
b) an object with mass ($m_o\neq 0$) ***at rest***, $p=0$, $p^2c^2=0$ and $E=E_0=m_oc^2$ which is its **rest energy.**
c) an object with mass($m_o\neq 0$) ***moving at*** **very high speed** (*i.e.* $\upsilon\sim c$ *and* $\gamma>>1$) , it comes out that $p^2c^2=(\gamma\, m_0\, \upsilon)^2\, c^2\sim\gamma^2\,(m_0\, c)^2\, c^2>>m_0^{\,2}*c^{\,4}\sim(m_0*c^2)^2$. So, $pc>>m_0*c^2$ (= E_o) and practically $E\approx p*c$. So, when a particle has a " **speed close** to *c* " ***it behaves like radiation (or a*** **wave**.)

5.CONSERVATION OF LINEAR MOMENTUM AND RELATIVISTIC MASS

-As mentioned previously, the postulate about "covariance" requests that the conservation principles hold on for any inertial frame. Let's see how the request for "covariance" on the principle of linear momentum conservation reveals relativistic effects on the inertia of objects by referring to the following experiment: *Two balls "1" and "2" with equal mass ($m_{1-0}=m_{2-0}=m_0$ as measured at rest) are compressed on the ends of a mass less spring and tied by a light thread so that the spring cannot expand. Once one cuts the thread, the ball "1" moves left side and the ball "2" moves right side. The set is initially at rest versus the inertial frame S'. Consider that the frame S' moves at a " high velocity $\vec{v}$ " versus another inertial frame S.*



The principle of linear momentum conservation is written as $\vec{P}_{sys}^{state\_A}=\vec{P}_{sys}^{state\_B}$ (35)

In the frame S', this relation gives $(m_{1-0}+m_{2-0})*0=\ m_{1-0}*\vec{u}_1'+\ m_{2-0}*\ \vec{u}_2'$ (36)
and by projecting on Ox' , one get $0=m_{1-0}(-u_1')+m_{2-0}(u_2')=-m_0*u_1'+m_0u_2'$

$$-u_1'+u_2'=0\ \ or\ \ u_2'=u_1'\equiv u'\ (equal\ magnitude).$$

So, the velocities of two balls in S'-frame are: $u_1'=-u'$; $u_2'=+u'$ (37)
Note: The magnitudes of u_1' *and* u_2' *are much smaller than the speed* υ *of frame S' vs. frame S.*

- Before writing the linear momentum conservation in frame S, one should remember that while Galileo transformations would give $x = x' + v * t$ and $u = u' + v$, *due to high speed motion of S'*, on must apply Lorentz transformations $x = \gamma(x' + v * t)$ and $u = \frac{u'+v}{1+\frac{v}{c^2}u'}$ in frame S. So, by applying Lorentz expression for the velocities, one get : $u_1 = \frac{-u'+v}{1-\frac{v}{c^2}u'}$ (38) and $u_2 = \frac{u'+v}{1+\frac{v}{c^2}u'}$ (39)

In frame S, the relation (35) is written as $(m_1 + m_2) * \vec{v} = m_1 * \vec{u}_1 + m_2 * \vec{u}_2$ (40)
and by projecting on Ox axe, one get $m_1 * v + m_2 * v = m_1 * u_1 + m_2 * u_2$ (41)
By rewriting this relation as $m_1 * (v - u_1) = m_2(u_2 - v)$

one get $m_1 * (v - \frac{-u'+v}{1-\frac{v}{c^2}u'}) = m_2(\frac{u'+v}{1+\frac{v}{c^2}u'} - v)$ or $m_1 * (\frac{v-\frac{v^2}{c^2}u'+u'-v}{1-\frac{v}{c^2}u'}) = m_2(\frac{u'+v-v-\frac{v^2}{c^2}u'}{1+\frac{v}{c^2}u'})$

Next, one get $m_1 * u' * \left(\frac{1-\frac{v^2}{c^2}}{1-\frac{v}{c^2}u'}\right) = m_2 * u' * (\frac{1-\frac{v^2}{c^2}}{1+\frac{v}{c^2}u'})$ which brings to $m_1 * \frac{1}{1-\frac{v}{c^2}u'} = m_2 * \frac{1}{1+\frac{v}{c^2}u'}$

One may note $\frac{v}{c^2}u' \equiv a$ (42) and write the last relation as $\frac{m_1}{m_2} = \frac{1-a}{1+a}$ (43) or $\frac{m_1^2}{m_2^2} = \frac{(1-a)^2}{(1+a)^2}$ (44)

By taking the squares of (38-39) and using label "a", one get $u_1^2 = \frac{(v-u')^2}{(1-a)^2}$ (45) $u_2^2 = \frac{(v+u')^2}{(1+a)^2}$ (46)

Then, $c^2 - u_1^2 = c^2 - \frac{v^2-2vu'+u'^2}{1-2a+a^2} = \frac{c^2-2ac^2+a^2c^2-v^2+2vu'-u'^2}{1-2a+a^2} = \frac{c^2-2vu'+\frac{v^2}{c^2}u'^2-v^2+2vu'-u'^2}{1-2a+a^2} = \frac{c^2-v^2-u'^2+\frac{v^2}{c^2}u'^2}{(1-a)^2}$

where one uses (42) to substitute $2ac^2 = 2\left(\frac{v}{c^2}u'\right)c^2 = 2vu'$ *and* $a^2c^2 = \left(\frac{v}{c^2}u'\right)^2 c^2 = \frac{v^2}{c^2}u'^2$. Similarly, one get

$c^2 - u_2^2 = c^2 - \frac{v^2+2vu'+u'^2}{1+2a+a^2} = \frac{c^2+2ac^2+a^2c^2-v^2-2vu'-u'^2}{1+2a+a^2} = \frac{c^2+2vu'+\frac{v^2}{c^2}u'^2-v^2-2vu'-u'^2}{1+2a+a^2} = \frac{c^2-v^2-u'^2+\frac{v^2}{c^2}u'^2}{(1+a)^2}$

One may factorize " c^2 " at left side, label " $\beta_1 = \frac{u_1}{c}$; $\beta_2 = \frac{u_2}{c}$ " and write those relations as :

$$c^2\left(1 - \frac{u_1^2}{c^2}\right) = c^2(1 - \beta_1^2) = \frac{c^2-v^2-u'^2+\frac{v^2}{c^2}u'^2}{(1-a)^2} \qquad (47)$$

$$c^2\left(1 - \frac{u_2^2}{c^2}\right) = c^2(1 - \beta_2^2) = \frac{c^2-v^2-u'^2+\frac{v^2}{c^2}u'^2}{(1+a)^2} \qquad (48)$$

Next, by dividing relation (48) by relation (47) on both sides and referring to relation (44), one get

$$\frac{1-\beta_2^2}{1-\beta_1^2} = \frac{(1-a)^2}{(1+a)^2} = \frac{m_1^2}{m_2^2} \qquad (49)$$

This last relation brings to $$m_1 * \sqrt{1 - \beta_1^2} = m_2 * \sqrt{1 - \beta_2^2} \qquad (50)$$

As the factor inside the square root is a number, the product present a mass and the relation (50) fits to the fact that rest masses of two balls in frame S' are equal($m_{1-0}=m_{2-0}=m_0$). So, one can write

$$m_1 * \sqrt{1 - \beta_1^2} = m_{1-0} = m_2 * \sqrt{1 - \beta_2^2} = m_{2-0} \qquad (51)$$

and get that $m_1 = m_{1-0} * \frac{1}{\sqrt{1-\beta_1^2}}$ $m_2 = m_{2-0} * \frac{1}{\sqrt{1-\beta_2^2}}$

This calculation shows that the special relativity request for " covariance " applied on the conservation of linear momentum brings to the result that the inertial mass of a "*relativistic object* " increases with its speed. So, if an object with mass is moving at a "*high speed* " versus a reference frame, one should use the relativistic mass expression :

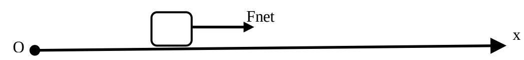
$$m = \frac{1}{\sqrt{1-\beta^2}} * m_0 = \gamma * m_0 \qquad (52)$$

6. A FORMAL WAY OF DERIVING THE ENERGY-MASS RELATIVISTIC RELATION

As mentioned previously, after upgrading Galileo transformations to Lorentz transformations, Einstein transferred all basic theorems of mechanics to special relativity. So, the work definition (53), the modern way of writing 2nd law (54) and the energy-work theorem (55), are valid in special relativity, too.

$dW_{Net} = F_{Net} * dx$ (53) $\qquad F_{Net} = \frac{dp}{dt}$ (54) $\qquad dE_K = dW_{Net}$ (55)

Now, let's refer to an object that, under the action of the net force F_{Net} along direction Ox, starts moving from rest ($v = 0$) and gets to a relativistic speed " $v = u$ ".



By substituting (54) in (53) one get $\quad dW_{Net} = \frac{dp}{dt} * dx = \frac{dx}{dt} * dp = v * dp$ (56)

Next, as $\quad d(v * p) = v * dp + p * dv \quad$ one get $\quad v * dp = d(v * p) - p * dv$ (57)

Then (55) can be written as $\quad dE_K = dW_{Net} = v * dp = d(v * p) - p * dv$ (58)

Now, one get the integral of both sides at (58) $\quad \int_0^{E_K} dE_k = \int_{v=0}^{u} d(vp) - \int_{v=0}^{u} pdv$ (59)

and $\quad E_K = u * p - \int_{v=0}^{u} pdv = u * mu - \int_{v=0}^{u} pdv = mu^2 - \int_{v=0}^{u} pdv = \gamma m_0 u^2 - \int_{v=0}^{u} \gamma m_0 v dv$

$$E_K = \gamma m_0 u^2 - m_0 \int_{v=0}^{u} \gamma v dv = \gamma m_0 u^2 - m_0 \int_{v=0}^{u} \frac{1}{\left[1-\left(\frac{v^2}{c^2}\right)\right]^{1/2}} v dv = \gamma m_0 u^2 - m_0 c^2 \int_{v=0}^{u} \frac{\frac{v}{c}}{\left[1-\left(\frac{v^2}{c^2}\right)\right]^{\frac{1}{2}}} d\left(\frac{v}{c}\right)$$

$E_K = \gamma m_0 u^2 - m_0 c^2 \int_{v=0}^{u/c} \frac{x}{[1-x^2]^{\frac{1}{2}}} dx \qquad$ As, $\quad \int \frac{xdx}{[a^2-x^2]^{1/2}} = -[a^2 - x^2]^{1/2} \quad$ for a = 1 one get

$$E_K = \gamma m_0 u^2 - m_0 c^2 \int_{v=0}^{u/c} \frac{x}{[1 - x^2]^{\frac{1}{2}}} dx = \gamma m_0 u^2 - m_0 c^2 * \left[-[1^2 - x^2]^{1/2}\right]_0^{u/c}$$

$$E_K = \gamma m_0 u^2 + m_0 c^2 * \left[[1 - x^2]^{1/2}\right]_0^{u/c} = \gamma m_0 u^2 + m_0 c^2 * \left([1 - (u/c)^2]^{\frac{1}{2}} - [1 - 0]^{\frac{1}{2}}\right)$$

$$E_K = \gamma m_0 u^2 + m_0 c^2 * \left([1 - (u/c)^2]^{\frac{1}{2}} - 1 \right) = \gamma m_0 u^2 + m_0 c^2 * \left(\frac{1 - (u/c)^2}{[1 - (u/c)^2]^{\frac{1}{2}}} - 1 \right)$$

$$E_K = \gamma m_0 u^2 + \frac{m_0 c^2 - m_0 c^2 * (u/c)^2}{[1 - (u/c)^2]^{\frac{1}{2}}} - m_0 c^2 = \gamma m_0 u^2 + \frac{m_0 c^2}{[1 - (u/c)^2]^{\frac{1}{2}}} - \frac{m_0 u^2}{[1 - (u/c)^2]^{\frac{1}{2}}} - m_0 c^2$$

$$E_K = \gamma m_0 u^2 + \gamma m_0 c^2 - \gamma m_0 u^2 - m_0 c^2 = \gamma m_0 c^2 - m_0 c^2$$

and finally, $\quad E_K = (\gamma - 1) m_0 c^2$ (60)

As $\quad E_{Tot} = E_K + E_{rest = internal\ en.} \quad$ one get $\quad E_K = E_{Tot} - E_{rest = int.en.} = \gamma m_0 c^2 - m_0 c^2$

From last expression one get $\quad E_{Tot} = \gamma m_0 c^2 = mc^2$ (61) $\quad$ and $\quad E_{rest = int.en.} = m_0 c^2$ (62)

7.SPECIAL RELATIVITY VERSUS CLASICAL MECHANICS

- Classical mechanics uses the ***material point model*** to study the motion of objects with mass. This model was introduced in mechanic "*as a theoretical operation that does make sense* " and *helps to solve problems* but it was *officially accepted in physics* because the related *results get confirmed experimentally*. The *criteria of experimental proof* took over the request " *for something that makes sense* " from the first steps of physic's developments. This criteria was used to verify the theory of special relativity, too. Note that it took about fifty years to confirm the relation $\boldsymbol{E = mc^2}$ experimentally and accept this theory officially.

- One has introduced the energy concept in mechanics (*changes of energy have a physical meaning*) via the Work - Energy theorem. Next, one has defined the *mechanical energy* "*ME* " as *ME = KE+PE* and the *total energy* of an object as $E_{object} = ME + E_{int}$ where E_{int} is the *sum* of "***all*** its ***internal energies***"(*thermal, molecular, atomic, nuclear, ….*) . If one attaches the reference frame to the object itself, then $KE = 0$ and if there is no external action on object $PE=0$. In this circumstances ME = 0 and the total energy of object is equal to its internal energy $E_{object} = E_{int}$. So, the *internal energy* is equal to the energy of an *isolated object at rest, i.e.* its rest energy, $E_{int} \equiv E_o$. The special relativity allows to calculate the internal energy of an object i.e. its ***rest energy as*** $E_{int}= E_o = m_oc^2$**.** One might figure out that the *rest energy* is converted in radiation energy if the whole mass of object is annihilated (*i.e. even its constituent elementary particles* e^-*,* p^+*, n.. are destroyed*). By defining 0-value for energy of "annihilated object", this relation allows to refer to absolute values of energy (not only to their differences) and this way "*upgrades"* the *parameter energy* to the same level as that of *parameter mass* in physics. In a way, the energy becomes a primary parameter because it applies for all nature objects (with mass & without mass - E.M. fields).

-One may rewrite relation (26) as $m=E / c^2$ and get $\Delta m=\Delta E/c^2$ (63)
Relation (63) tells that the mass of an object changes anytime there is a change of any form (*mechanical, thermal, chemical, electrical,.*) of its energy. Calculations based on expression (63) show that the amount of mass change "Δm" of considered objects during experiments in mechanics, thermodynamics, electricity or chemistry is smaller than measurements uncertainty and cannot be verified but it should be measurable during a nuclear reaction (*where* ΔE_{int} *is about* 10^6 *times larger than during a chemical reaction*).
In fact, there were the experiments in nuclear and elementary particle physics that confirmed the mass change as expected by relativity expression (63). During a nuclear reaction event, the total mass of products is smaller than mass of nuclide "*mother"* and the *amount of lost mass* Δm is converted to the energy of products in conformity to relation $\Delta E= \Delta m*c^2$.
The experiments on the "pairs production" (like an electron e- and a positron e+ *same mass but opposite sign electric charge*) in high energy radiation fields confirm the *conversion of energy into mass* following the relation $\Delta E= \Delta m*c^2$, too.
These experiments constitute the proof for ***reciprocal correspondence*** between ***mass and energy at (26).***

- The special relativity and quantum mechanics brought new theories and models in physics. What is the relation of these new theories with the classical physics? Here, one must keep in mind that all models and theories of classical physics are confirmed by experiment and they are valid for all phenomena in the range of their applications. So, a basic request for any new theory is : its models and its expressions must fit to those of classical physics when they apply to the range of validity of classical physics. The following calculations show that this request is fulfilled by special relativity expressions.

Example: A bullet with speed $\upsilon \sim 10^3 m/s$ is not a relativistic particle because $\beta = v/c \sim 10^{-5}$ or $\upsilon^2/c^2 \sim 10^{-10}$
and $\gamma = \frac{1}{\sqrt{1-(v/c)^2}} = [1-(v/c)^2]^{-1/2} \approx 1 + \left(\frac{1}{2}\right) * (v/c)^2 = \cdots \ldots \ldots \ldots .. = 1.00000000005$
Then, one gets $K = (\gamma - 1)m_oc^2 \cong \left(1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 - 1\right) m_oc^2 = \frac{v^2}{2c^2}\, m_oc^2 = \frac{m_ov^2}{2}$ (64)

or by neglecting term " $(\upsilon/c)^2 \sim 10^{-10}$ " $m = \gamma m_o \cong \left[1 + \left(\frac{1}{2}\right) * (v/c)^2\right] m_o \cong 1 * m_o = m_o$ (65)

and $p = mv = \gamma m_o v \cong \left[1 + \left(\frac{1}{2}\right) * (v/c)^2\right] m_o v \cong 1 * m_o v = m_o v$ (66)

- Note that there is a divergence of opinions about the mass. Several *authors neglect expression* (62) *assuming that mass "m" is an* ***invariant parameter*** *(i.e. the same in all inertial frames) and it does not make sense to talk about a* ***rest mass m₀***. This point of view is generated from the relation of gravitational attraction to the curvature of 4D space-time in the theory of general relativity. For these authors, there is a fixed amount of mass inside an object with mass and it does not change, no matter what is the speed of object. The *equivalence* between mass and energy is an important source of discussions inside this debate.

In this course, one considers the *inertial mass* as a *primary* parameter in physics and makes use of expression (62). This point of view is supported by the following arguments:

a) Physics has defined the mass as ***the parameter that measures the property of inertia of objects*** and all classical mechanics results are based on this definition. When introducing his law for universal gravitation, Newton assumed that *gravitational mass* could be distinct from *inertial mass*. But, later on, his own experiments with simple pendulum (see Benson p.268) showed that they are identical if one chooses the value for universal gravitation constant $G = 6.67408 * 10^{-11}\ m^3\ kg^{-1}\ s^{-2}$. Here it is interesting to mention that the human sensation is more tented versus the notion of " *gravitational mass as an amount of material inside object* " but in physics, the *mass is introduced and used as a measure of inertia* of object.

b) The graph of relativistic momentum in fig. 6 shows, that one has to ***apply larger action*** ($F_{ext} \sim \Delta p$, *i.e. the graph slope*) for the same change of particle speed "say $\Delta\upsilon = 0.2c$" in speed range "*0.6c to 0.8c"* compared to range "*0.2c to 0.4c"*. From the definition of inertial mass ("*as opposition of object vs. change of its velocity* ") this means that the inertia of object (i.e. *its inertial mass*) does increase with its speed.

c) The normal function of synchrocyclotrons (accelerator of relativistic particles) requires a frequency tuning to cope with the inertial mass change as predicted by expression (52).

-One needs to present all the formalism of the general relativity so that one can discuss the arguments for the model of constant gravitational mass. At this point, one may get an adjustment between the two models by simply substituting the rest mass "m_o" by gravitational mass "m_G " and labelling this last one as "m". Then, as $m_o = m_G = m$, one removes the relation (28,52) and the relations (26',26",27) are written as:

$E_0 = mc^2, E = \gamma mc^2$, $\vec{p} = \gamma m\vec{v}$ and relations (30,34) $K = (\gamma - 1)E_0$, $E^2 = (pc)^2 + E_0^2$ do not change.

Note that in all those last five expressions one refers to the mass "m" as a invariant or constant parameter.