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REMEMBER

- The **black body** is an ideal object that absorbs & emits all EM radiations.
One refers to this model for the study of thermal radiation in equilibrium at a controlled temperature**.**
The ***colour*** of thermal radiation (i.e. *blackbody radiation*) depends ***only on the temperature of its "source".***

- The ***energy density*** of **a *radiation u(T)***[J/m^3] is the total amount of EM radiation energy contained into a volume of ***1m^3***. The ***spectral*** *energy density* ***u(T)$_\lambda$*** [J/m^4] informs about the distribution of **u(T)** through different wavelengths. One can get **u(T)$_\lambda$** from the spectrum of the thermal radiation.

-The **Wien's first law** relates the *wavelength at maximum intensity* and the *temperature* at the source of the thermal radiation spectrum

$$\lambda_{\max} * T = 2.898 * 10^{-3} [mK]$$

- Planck hypothesis about *minimum amount of transferred energy* $\varepsilon = h * f; _where_ f = c/\lambda$ with $h = 6.626 * 10^{-34}\, J * s$ produced the expression $u_\lambda(T) = \dfrac{8\pi hc\lambda^{-5}}{e^{hc/\lambda kT} - 1}$ which fits to experimental records.

-***The photoelectric effect is the emission of electrons by a metal when a beam of light falls on it.***
One can find the maximum speed of ejected electrons by using the stopping potential "V_0" and expression

$$K_{Max} = \frac{1}{2} m \upsilon_{\max}^2 = eV_0$$

-A. Einstein explained the basics of photoelectric effect by introducing the quantum model of light:

Radiation can be modelled as a collection of photons, each containing the amount of energy ε = h*f .
During a collision "*photon - free e-*", it happens an *instantaneous transfer of energy* "***ε***" to the electron. One part of "***ε***" absorbed by "*e-"* goes for the work function, one part for possible *interactions* inside metal; the rest goes for kinetic energy of photoelectron. If there is no internal *interactions, K = K_{max} and*

$$\varepsilon = \varphi + K_{max}^{el}$$

- Compton effect is the scattering of ***X-ray photons*** by ***free electrons(or loosely bound)***. The scattered radiation contains photons at a wavelength $\lambda' \neq \lambda_{incident}$; *$\lambda'$ value depends only on the scattering angle*.

This effect is experimental proof that x-rays behave as photons and they have a linear momentum.

$$\lambda' - \lambda = \left(\frac{h}{m_0 c}\right)(1 - \cos\theta) \qquad (h/m_{0*}c) = 0.00243 \text{ nm (Compton wavelength.)}$$

- The light is *emitted* during the *transition of a quantum system*(atom, molecule..) *from* one level of *higher energy* to a level of *lower energy*. The light can be *absorbed by an inverse transition*. The frequency and wavelength of emitted (or absorbed) light related to a transition $1 \Leftrightarrow 2$ can be calculated by the relations

$$\varepsilon = E_2 - E_1 = hf; \qquad where\ f = c/\lambda$$

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THE DUALITY WAVE - PARTICLE OF LIGHT

- A brief summary of historical records about the physical models used for the light shows that:
  a) From the antiquity till 19th century one considered that **light is** a **stream of tiny particles**.
  b) During the 19th century till the beginning of 20th century, the physicists used extensively the **wave model for light**. The experiments (Young , Fresnel…) and the theory (Maxwell, Huygens..) provided a full set of proofs for the wave model of light.
  c) By the end of 19th century and during the first 30 years of 20th century, appeared a number of experiments (thermal radiation, photoelectric effect, Compton effect, line spectra from atoms, etc), that have explanation only in the frame of a **quantum** model for which the light is constituted by ***photons*** i.e. "***bubbles of*** light ***energy***" that behave as " ***particles*** " with ***linear momentum***.
  So, it was normal to ask: "What is the **true** nature of light; ***wave or particle***?"

- Note that it was not easy to get an answer for this question because physics is an experimental science and **there was experimental evidence for the two models of light**.

Initially, the physicists tried to solve this puzzle based on the following arguments:

a) At ***low frequency*** range of electromagnetic radiations, a ***single photon*** contains extremely ***small amount of energy*** (***ε_{ph} = h*f***). In this range (***radio & TV waves***) the measurements deal with a very **large number** (*billions*) of **photons;** one *cannot isolate and observe the behaviour of a single photon. So, the observed* ***wave*** *behaviour of EM concerns the* ***collective behaviour*** *of a big number of photons*.

b) At ***high frequency*** range of electromagnetic radiations, a ***single photon*** has ***high energy*** (***ε_{ph} = h*f***). When performing measurements in this range (***X or γ-rays***), one "***deals with single events"*** and one is *more realistic* when judging about the *true nature* of electromagnetic radiation. Experiments in this range of frequencies confirm the particle (***photon***) **behaviour** of radiation. So, this should be the true nature of all E&M radiation and, as a part of it, the true nature of visible light, too.

- Hence, physicists thought to look for ***the ultimate proof*** of particle model of E&M radiation in the region of ***visible light*** where both light models (***particle*** and ***wave***) apply successfully (as *there is experimental proof for both of them*). The essential point of view for all these experiments was the following:
Assuming that the wave behaviour concerns only a large number of photons, the interference and diffraction patterns should disappear if one sends "each photon separately " at the input of slits.

So, one checked this hypothesis by **sending** *visible light* **photons "one by one"** (*by using extremely low intensity beams***)** towards slits in diffraction experiments(*with single slit or two slits*). But, when observing the screen after a long time they found always the same diffraction-interference pattern as if many photons were sent simultaneously. These experiments brought to the conclusion that ***wave behaviour of light is not a feature that concerns only a big number of photons***. ***Even a single photon has wave characteristics.*** Consequently, the ***duality particle – wave comes out as an intrinsic feature of light***.

- This type (*dual*) of behaviour appears clearly when one considers a ***light photon*** in the frame of the theory of ***special relativity***. When applying the expression for total energy of a relativistic particle

$$E^2 = p^2c^2 + (m_0c^2)^2 \qquad (1)$$

in the case of ***a single photon*** ($m_0^{ph} = 0;\quad E^{ph} = \varepsilon = h * f;\quad f = \frac{c}{\lambda}$) one get

$$E_{ph}^2 = p_{ph}^2 c^2; \rightarrow p_{ph} = E_{ph}/c; \rightarrow p_{ph} = h * f/c = h/(c/f) = h/\lambda$$

So $$p_{ph} = h/\lambda \qquad (2)$$

This relation associates (*or attaches*) *to a wave with wavelength "λ"* a *particle with linear momentum "p_{ph}"*. Note that both characteristics (**λ** of a *wave* and **p_{ph}** of a *particle*) belong to the **same physical object** (light).

- It's true that many natural phenomena are explained **in full inside** *particle model* and some others are explained **in full inside** the *wave model* but the ***light behaviour cannot be explained in full inside only one of those models***. In some experiments its behaviour is explained by the **wave model** and in some others by the **particle model**. Besides, some of ***light characteristics cannot be supported by any of these two models***:
  - Light behaves as a wave "**with no need for a specific propagating medium**"!!
  - Light behaves as a particle which "**speed does not depend on the frame where one measures it**"!!

Conclusion; "There is ***no sense*** to talk about the ***true nature*** of light". For this reason physics simply uses:

a) the **wave model** to explain diffraction and interference phenomena and
b) the **particle model** to explain the **absorption, emission and scattering of light.**

THE MATTER WAVES

-Bohr's model introduced the "**allowed circular orbits"** (fig 1) to *explain the origin of line spectra* of hydrogen and other one-electron ions. But, this model has three weak points:

a) It *does not give any justification about the reason why only those orbits are allowed.*
b) It does not work for atoms with more than one electron.
c) It does not offer any way for *calculating the intensities of spectral lines related to transitions.*

Actually, the **major contribution** of this model is the **introduction of quantification into different parameters of atomic models, even though in an unjustified way**.

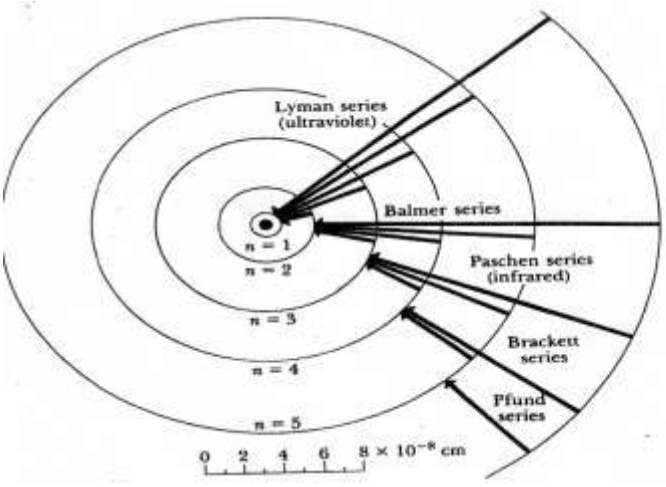


Fig 1

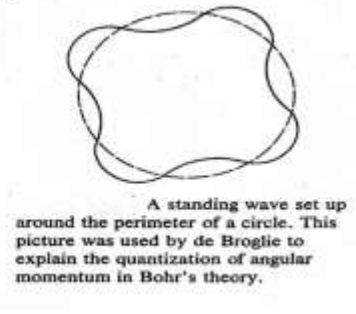
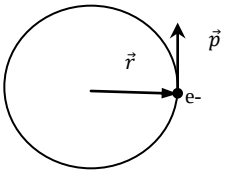

A standing wave set up around the perimeter of a circle. This picture was used by de Broglie to explain the quantization of angular momentum in Bohr's theory.

Fig (2)

- The next important development in quantum theory was the introduction of the *duality behaviour into the models used for the matter*. In 1924, Louis de **Broglie** introduced the postulate:
" **The dual behaviour** *" wave-particle "* **is valid for objects *with* mass**, too.

D**e Broglie *principle*** states that:
To any **particle** with **linear momentum *p = mv*** is **associated a wave** with wavelength λ
given by relation $\lambda = h / p$ (3)
Note that relation (3) is the same as relation (2) but it is written and read in a reverse order.

- The first **proof** of validity for this principle was the justification of "**allowed orbits"** in Bohr's model. To get the expression for energy levels, Bohr *assumed* that the **magnitude** of **angular momentum** " ***l*** " ($\vec{l} = \vec{r} \times \vec{p}$) of electron in an allowed orbit with principal quantum number " *n* " must fulfill the condition



$$l_n = r_n p = n\frac{h}{2\pi}; n = 1,2,3 \ldots \quad (4)$$

Bohr used this condition simply as a mathematical request to get an expression that fits to the recorded line spectra of hydrogen but he did not justify his assumption from physic´s point of view. If one applies de Broglie relation (3) for the linear momentum of an electron at (4) one get the expression (5)

$$r_n p = r_n \frac{h}{\lambda} = n\frac{h}{2\pi}; \rightarrow\rightarrow\rightarrow 2\pi r_n = n\lambda; \quad n = 1,2,3 \ldots \quad (5)$$

The expression (5) is similar to a requirement for ***wave resonance*** in a string or in a pipe length. So, the de Broglie postulate provides an justification based on physic's models for the arbitrary **allowed orbits:** " Only ***orbits that fit an integer number of e⁻ wavelength on their circumference are allowed*** (see fig.2)". This explanation of Bohr's orbits is based on the idea that <u>*there is a wave associated to the electron.*</u>
Note that the model of discrete orbits allows to calculate the wavelengths of atomic line spectra which are observed experimentally. This way the line spectra are an <u>indirect proof</u> of wave behaviour of electron i.e. the existence of " e- " related waves.

<u>ELECTRON DIFFRACTION</u>

- The explanation of "allowed orbits" in Bohr's model was an indirect proof for existence of ***matter waves***. The direct proof needed a diffraction or interference pattern produced by those waves. In 1926, Davisson and Germer recorded the diffraction pattern of "e-" waves. In this experiment, a heated filament produced electrons which were provided a linear momentum by an electric potential difference V (fig 3). The beam was directed on a **Ni target** and one measured the number of " *scattered electrons* " along different angles.

- Depending on the behaviour of electrons, two different results might appear:

a) If electrons are *only* ***particles***, after collision with regularly arranged atoms in the crystalline array of Ni, they should be ***scattered uniformly*** in space.

b) If the electrons behave as **waves**, after collision with regularly arranged atoms in the crystalline array of Ni, they should be scattered "as *diffracted* " by obeying to the ***diffraction-interference*** rules for waves. In this case one must record a **greater number of e-** scattered **along** some ***specific space directions***.

One might remember that the diffraction-interference pattern of a wave is produced only if its wavelength λ is *comparable to* dimensions of diffracting structure, in this case to "*distance D between Ni atoms* ".

Davisson and Germer used the electric voltage of an electron gun (see fig.3) to tune the " *wavelength of electrons*". By use of an electric potential ***V***, they provided to electrons the kinetic energy

$$K = \frac{mv^2}{2} = \frac{m^2v^2}{2m} = \frac{p^2}{2m} = eV \qquad \text{and} \qquad \text{the linear momentum} \qquad p = \sqrt{2meV} \qquad (6)$$

Then, based on the expression (3), the corresponding wavelength of ***e-*** wave would be

$$\lambda = h / p = h / \sqrt{2meV} \qquad (7)$$

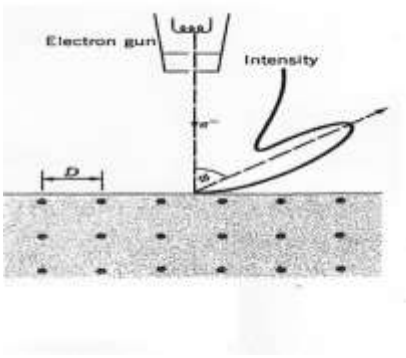


Fig 3

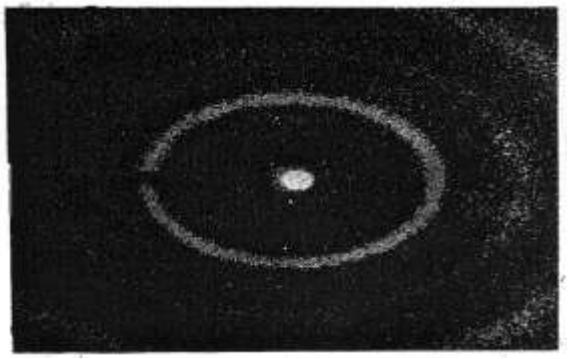


Fig 4

They could tune λ values of *e- wave* by changing the voltage value ***"V "***. The " *inter-slit distance* " for Ni crystal is D = 0.215nm. So, they had to apply an voltage V that gives λ-values of the same order. When this requirement was fulfilled, they observed that the number of scattered electrons was larger along several particular directions of space (see fig.3 and bright circles in fig.4).
This was the direct **experimental proof for the wave behaviour of electrons.** Later on, one could prove the ***dual nature*** *(particle-wave)* for neutrons and protons by experiments, too.

SHREDINGER'S WAVE EQUATION

- By the second decade of 20th century, there was ***experimental proof*** *for* the ***dual nature*** *of* ***light*** and ***subatomic particles***. Also, there were **theoretical models** for the ***two facets of light*** and for ***particle behaviour of matter*** but here was ***no theoretical model*** for the ***wave behaviour of particles with mass***.

- Erwin Schrödinger filled this gap. One might remember that, *despite the different physical meaning of their "displacement", the mechanic waves and light (or E.M.) waves obey to the same* ***mathematic*** *form of* ***wave equation***. So, like any other wave phenomena, the " ***matter waves*** or ***De Broglie waves ",*** *no matter what is the meaning of their " displacement - y",* must obey to the same **wave equation; i.e. (***in 1D space)*

$$\frac{\partial y^2}{\partial x^2} = \frac{1}{\upsilon^2}\frac{\partial^2 y}{\partial t^2} \qquad (8)$$

"y" is the '*displacement* ' of matter wave; ' υ ' is its propagation speed
"x" is the space variable; " t " is the time variable

- As the **selection rule** of "allowed orbits" in Bohr's model is due to a ***resonance condition*** for e- waves, the " *displacement-y*" (i.e. their function) of these waves must be expressed as ***product*** *of two* ***functions:*** *one* ***space-dependent*** and one ***time-dependent*** (*like 1D resonant waves of string* $y = 2A\sin(kx) * \cos(\omega t)$ *).* So, one can write the wave function of an ***e*** $^-$ ***bound in an atomic orbit*** in a 1D spatial model in the form

$$y(x,t) = \psi(x) * \cos(\omega t) \tag{9}$$

where the function ψ(x) is a function of space coordinates.

In the following, one uses the wave equation (8) to get a way for definition of space-dependent function.

Second order derivatives of (9) are: $$\frac{\partial y}{\partial x} = \frac{d\psi(x)}{dx}\cos(\omega t); \frac{\partial^2 y}{\partial x^2} = \frac{d^2\psi(x)}{dx^2}\cos(\omega t) \tag{10}$$

and $$\frac{\partial y}{\partial t} = -\omega\sin(\omega t)\psi(x); \frac{\partial^2 y}{\partial t^2} = -\omega^2\cos(\omega t)\psi(x) \tag{11}$$

By substituting the expressions (10,11) into equation (8) one gets $$\frac{d^2\psi(x)}{dx^2} = -\frac{\omega^2}{\upsilon^2}\psi(x) \tag{12}$$

As **e**$^-$ energy is $E = K + U = \frac{mv^2}{2} + \mathrm{U} = \frac{m^2v^2}{2m} + U = {p^2}/{2m} + \mathrm{U}$ and $p^2 = \sqrt{2m(E-U)}$ (13)

one get $$\left(\frac{\omega}{v}\right)^2 = \left(\frac{2\pi}{Tv}\right)^2 = \left(\frac{2\pi}{\lambda}\right)^2 = \left(\frac{2\pi}{h/p}\right)^2 = \left(\frac{p}{h/2\pi}\right)^2 = \frac{p^2}{\hbar^2} = \frac{2m(E-U)}{\hbar^2} \tag{14}$$

One usually uses $\hbar = \frac{h}{2\pi}$ *known as the reduced Planck constant in quantum mechanics expressions.*

Then, by substituting (14) at (12) one gets $$\frac{d^2}{dx^2}\Psi(x) = -\frac{2m(E-U)}{\hbar^2}\Psi(x) \tag{15}$$

or $$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Psi(x) = (E-U)\Psi(x) \quad \text{and} \quad \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \mathrm{U(x)}\right]\Psi(x) = E\Psi(x) \tag{16}$$

which is written mostly as $$\mathrm{H}\Psi(x) = E\Psi(x) \tag{17}$$

"U(x)" is the potential energy function, "H" is the Hamiltonian operator, "E" is the energy of quantum system.

The equation (15,16,17) is known as ***one dimensional time-independent Schrödinger wave equation***.

When applied for atoms with a single bound electron (like H atom), it has solutions ψ_n *only for a set of energy values* E_n *that fit to Bohr's expression. Note that this equation provides the energy levels and the corresponding* ψ_n *wave functions even for atoms with many electrons.*
Schrödinger equation was the decisive step in the development of quantum mechanics.

- What does represent the wave function of a quantum particle? Max Born provided the physical **meaning** for the **wave function** ***attached to a quantum particle***. One can get it by interpreting the relation between the *light wave* ***intensity*** and the ***number*** of ***photons*** in the frame of ***1- slit diffraction.*** At first, one should remember that the ***intensity*** of *light wave at a* ***point*** is proportional to the ***number of photons at this location,*** i.e. it is ***proportional*** to the ***probability*** of finding a photon ***at this point***. As the light ***intensity*** is proportional to the "***square*** *of* ***displacement*** " of *light wave* (i.e. $\mathbf{E^2(x,y,z)}$ ***function*****)**, it comes out that the ***square of*** *light wave* ***displacement function*** at a location is *proportional* to the ***probability*** of finding *a photon* there. The single slit diffraction pattern in fig. 5 visualizes the relation between light intensity "***I***" and density of photons on screen. As governed by same rules, the diffraction of *quantum waves* associated to *electrons* by a narrow slit produces a similar pattern as that of photons (light wave) diffraction (Fig. 5).

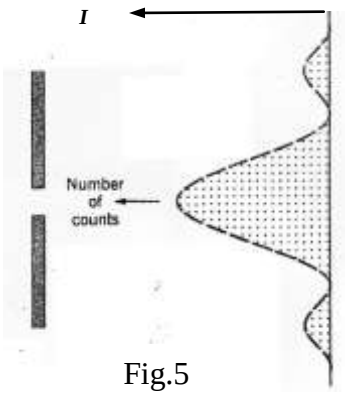


Fig.5

Based on this correlation, Max Born ***postulated*** that the *square of the wave function* ψ^2**(x,y,z)** of a **quantum particle** is proportional to the probability of finding this particle at a given location **(x,y,z)** of space. Next, he determined the *probability of finding the particle within an infinitely small* space *volume dV* as

$$\psi^2(x, y, z)\,\mathbf{dV} \qquad (18)$$

and the **probability to find the particle somewhere within the whole space** as

$$\int_{-\infty}^{+\infty} \Psi^2(x,y,z)\,dv = 1 \qquad (19)$$

- The expression (19) is known as the *normalization condition* and the function $\psi^2(x,y,z)$ represents a *density of probability* in the frame of the probability theory. Accordingly, the ***quantum wave function*** $\psi(x,y,z)$ contains ***a probabilistic information*** and the *language of quantum mechanics is a probability language*. So, there is no sense to talk for the exact position of a quantum particle but it does make sense to say that " it is *more likely to observe the particle* at locations where $\mathbf{\psi^2(x,y,z)}$ is larger ".

- The wave equation (15-17) contains only one space coordinate and its solutions are *wave functions* of *one coordinate*. When studying a problem in 3D space, one deals with **three *dimensional wave equation*s** and the solutions are ψ-functions of three variables *(x,y,z)*. The shapes of the "**probability clouds**" in Figure 6 correspond to the graphs of 3D functions $\psi_n(x,y,z)$ in the case of H-atom. They contain the circular Bohr orbits **and** have the same number of nodes as predicted by de Broglie rule for the " *resonant e- waves* ".

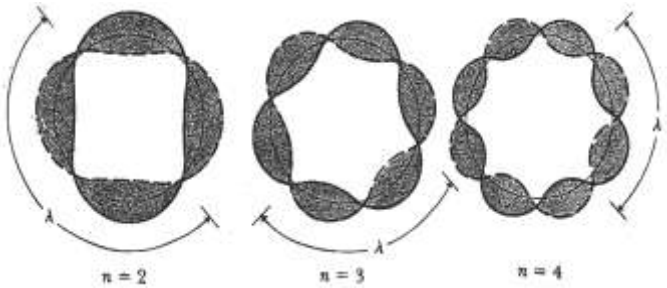


Fig.6 *Wrapping of 3D standing waves around Bohr's orbits. So that* e^- *standing wave fits to a given circular Bohr's orbit, the circumference length of this orbit must be equal to an integer number of electron wavelength. The number of hosted* λ *is equal to quantum number n (1,2,3,4...) of* e^- *orbit.*

THE HEISENBERG UNCERTAINITY PRINCIPLE

-Q*uantum mechanics* is based on the postulate that both *light and matter have dual behavior* and the square of wave function $\psi^2(x,y,z)$ gives the probability of finding a *quantum particle* at a given position. But, a *particle* is localized at a position of space while a wave is not. How to overcome this discrepancy?

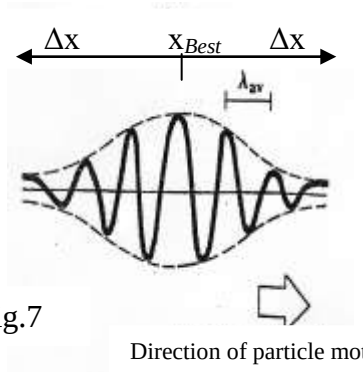


Fig.7

Fig.7 shows a **<u>wave packet</u>** which *has both wave and particle properties*. The regular spacing λ_{av} between consecutive maxima is a wave characteristic and it does have a *limited extension* $2\Delta x$ in space (*like a particle)*, too. To understand what way this wave packet is built, one might remember the superposition of ***two sound waves*** with *close* values of wavelengths λ,λ' that produces the sound *beats*. Actually, the shape of a sound *beat's wave* contains *many such wave packets*. One may show (*by calculation*) that, by increasing the *number* of *superposed waves* with very close λ-values, the result forms a ***single wave packet*** like this in fig.7; λ_{av} is the ***average*** value of all superposed wavelengths.

- The fact that the ***wave packet contains many wavelengths*** means a certain a ***spread*** $\Delta\lambda$ around λ_{av}. This spread brings automatically (*due to relationship* $p = h / \lambda$) the existence of a *related uncertainty* Δp for its linear momentum p. *The* ***uncertainty principle*** *of* ***Heisenberg*** *states that: The absolute* ***uncertainties*** *of* ***position*** *and* ***linear momentum*** *of a quantum particle are* ***interdependent****. For a quantum particle in 1D space, if one measures* ***simultaneously*** *"**x**" and "**p_x**", their absolute uncertainties obey to the condition*

$$\Delta x * \Delta p_x \geq h \qquad (20)$$

-This means that, no matter what is the perfection of experimental set up, at the best, one may expect only to get results that provide $\Delta x * \Delta p_x = h$ and there is no sense to look for $\Delta x * \Delta p_x < h$. One <u>*cannot measure* ***simultaneously***</u> both the position and the linear momentum of <u>*a quantum object with an* ***arbitrary precision***</u>. So, let's assume that one measures ***simultaneously*** the *position* and the *linear momentum* of a **quantum particle** and finds out the uncertainties Δx, Δp_x. Next, by changing the technique of measurement, one *improves the precision* for position measurement (Δx *decreases*). The relation (20) tells that the *precision* for *linear momentum* measurement will *deteriorate* (Δp_x *increases*). This comes as a ***fundamental restriction imposed by nature*** and there are no experimental ways one may skip it.

-It is important to mention that Heisenberg found this principle by analyzing the process of measurement for quantum particles. Here it is one example that allows to get to this principle. Consider the diffraction of a beam of electrons at same velocity(i.e. same linear momentum p) by a single slit (fig 8). Its *diffraction pattern is the same as that of a monochromatic light* with corresponding wavelength ($\lambda=h/p$). This means that one would record a minimum number of diffracted electrons along direction $\theta=\theta_{s=1}$ that corresponds to the " *first minimum of single slit diffraction* " and just a few electrons beyond it. One might estimate that, at the ***slit output***, the y-coordinate of a diffracted electron has an uncertainty $\Delta y \approx a/2$ and find out that,

$$\sin\theta = \lambda / a \rightarrow \sin\theta \cong \lambda / 2\Delta y \rightarrow \Delta y \cong \lambda / 2\sin\theta \qquad (21)$$

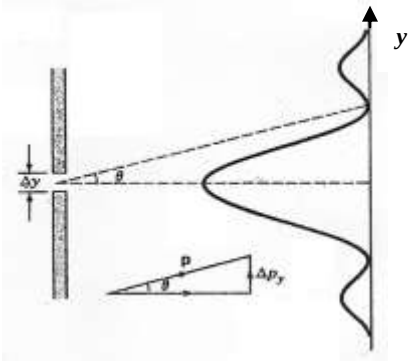


Fig 8

Note that linear momentum of e- has the uncertainty $\Delta p_y \approx p\sin\theta$ (fig 8) if it falls inside of *central maximum*. Meanwhile, some electrons fall out of the central maximum. This means even larger uncertainty for the linear momentum of an electron that pass through the slit. In other terms, at the output of slit, the uncertainty of linear momentum for electron is ***at least " psinθ"***. One can express this situation by the mathematical condition

$$\Delta p_y \geq p\sin\theta \geq \frac{h}{\lambda}\sin\theta \qquad (22)$$

By multiplying side-by-side the two expressions (19) and (20) one finds out that

$$\Delta y * \Delta p_y \geq h/2 \qquad (23)$$

***Note**: The precise mathematical expression for uncertainty principle used in quantum theory is written as*

$$\Delta y * \Delta p_y \geqq \hbar/2 \qquad (24)$$

$\hbar = h/2\pi$ and ***Δy, Δp$_y$*** *stand for the* ***standard deviations*** *of position and linear momentum components.*

As *this course* does not consider the details of quantum mechanics, we will refer to the expression (20) which means a *more conservative restriction* (*4π times larger*) for the correlated uncertainties. So, when solving quantum mechanics problems in this course, one may use the Heisenberg principle for the components of coordinate and corresponding linear momentum as

$$\Delta x * \Delta p_x \geq h \qquad \Delta y * \Delta p_y \geq h \qquad \Delta z * \Delta p_z \geq h$$

- The principle of uncertainty applies to all other couples of " *conjugated quantum variables* ", too.
For example, the ***uncertainties of measurement*** for ***time and energy*** are related similarly

$$\Delta E * \Delta t \geq h \qquad (25)$$

(or more precisely $\Delta E * \Delta t \geqq \hbar/2$)

One may derive expression (25) from (20) by referring to a *free* (*i.e. no potential energy*) *quantum non relativistic particle* moving at " *constant speed υ* " along Ox axe. Its total energy versus frame Ox is $E = E_0 + K$. So, one get $\Delta E = \Delta E_0 + \Delta K = \Delta K$ ($\Delta E_0 = 0$ because its rest energy does not change).

As $K = {}^1\!/_2\, mv^2$ and $\Delta K = {}^1\!/_2\, m\Delta(v^2) = {}^1\!/_2\, m(2v\Delta v) = v\Delta mv = v\Delta p$ (26)

So, when referred to motion along Ox, one get $\Delta E = \Delta K = v\Delta p_x$ (27)

Next, as $t = x / v$ one get $\Delta t = \Delta x / v$ (28)

By multiplying both sides of (27) and (28) one get $\Delta E * \Delta t = (v\Delta p_x) * (\Delta x / v) = \Delta x * \Delta p_x \geq h$

- There are two important consequences that come out from expression (25);
  a) If one wants *more precise* energy measurements (*i.e. to decrease ΔE*) for a ***quantum object*** (*molecule, atom, electron...*)**,** one must increase the observation time (*i.e. increase Δt*).
  b) The **lifetime τ_n** of an atomic **energy level E_n** is the ***average time*** that an electron spends at this level *before* leaving it for a *natural transition* towards lower energy levels. Even though this time may get values from 0 to infinity (Fig.9), the measurements show that, for " *excited* " states, the *average lifetime* has values $\tau_n \sim 10^{-6} \div 10^{-9}$s (it can be as long as 10^{-3}s at "*meta-stable*" quantum states).

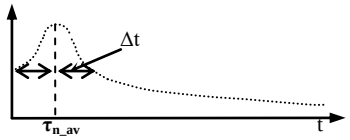


Fig. 9

As one may see from fig.9, the ***average*** value of τ_n and ***uncertainty*** Δt_n of the time an electron spends at excited quantum level E_n have similar values. So, in ***uncertainty*** calculations for energy levels, one can simply take $\Delta t_n = \tau_n$.

Then, from (25), one can find out that the ***uncertainty*** of energy level E_n is $\Delta E_n \geq h / \Delta t_n \geq h / \tau_n$
Note that, for a fundamental level of energy $\tau_1 = \infty$ and relation (25) gives $\Delta E_1 = h/\infty = 0$.

During a spontaneous transition from $\boldsymbol{E_n}$ to $\boldsymbol{E_1}$, the quantum system emits one photon with energy $\varepsilon_{photon} = E_n - E_1$. The energy of this photon has a "*minimum possible*" value of *uncertainty*

$$\Delta\varepsilon_{photon} = \Delta E_n + \Delta E_1 = \Delta E_n + 0 = \frac{h}{\Delta t_n} = h/\tau_n \qquad (29)$$

This means that, in general, the uncertainty in energy of an emitted photon is $\Delta\varepsilon_{photon} \geq h/\tau_n$.

As the frequency of this photon is $\boldsymbol{f = \varepsilon_{photon} / h}$, one can find out the uncertainty of frequency for the emitted wave as

$$\Delta f = \Delta\varepsilon_{photon} / h \geq (h/\tau_n)*1/h \geq \boldsymbol{1/\tau_n}. \qquad (30)$$

This relation shows clearly that the emitted wave has a *sharply defined* frequency *(small* $\boldsymbol{\Delta f}$ *)* if the upper *level* ($\boldsymbol{E_n}$) of transition has a long *life time (large* $\boldsymbol{\tau_n}$ *value).*

If the transition happens between two exited energy levels, say from $\boldsymbol{E_i}$ to $\boldsymbol{E_k}$, then, the emitted photon has the energy $\varepsilon_{photon} = E_i - E_k$ and the expressions (27, 28) become

$$\Delta\varepsilon_{photon} = \Delta E_i + \Delta E_k = \Delta E_i + \Delta E_k \geq \frac{h}{\Delta t_i} + \frac{h}{\Delta t_k} = \frac{h}{\tau_i} + \frac{h}{\tau_k} = h\left(\frac{1}{\tau_i} + \frac{1}{\tau_k}\right) \qquad (31)$$

Subsequently, $\Delta f_{ik} = \Delta\varepsilon_{photon} / h \geq h\left(\frac{1}{\tau_i} + \frac{1}{\tau_k}\right)*1/h \geq \frac{1}{\tau_i} + \frac{1}{\tau_k}$ (32)

As a general rule, the ***spectral lines*** due to transitions between energy levels with ***long lifetime*** $\boldsymbol{\tau_n}$ are ***sharp*** and vice versa.